A Model of Value-Added Taxes

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1 Introduction

1.1 Why is the VAT different?

In any model with informality and taxation, there will exist an inefficiency associated with the amount of output produced by medium-sized firms that constrain their output in order to remain unobserved by the tax authorities. In the presence of the VAT, though, there is an important distinction: informality in every stage of the production chain is determined in equilibrium by the monitoring thresholds on *all* stages of the production chain. That is, the size of the informal sector in the upstream sector is determined by the demand for the intermediate good, which depends on the monitoring threshold in the downstream sector $\bar{x}$. Analogously, the size of the informal sector in the downstream stage is determined by the supply of informal intermediate goods in the upstream point of the production chain, which depends on its monitoring threshold $\bar{y}$. Therefore, through equilibrium prices, varying the monitoring threshold on either stage of the production chain will affect informality on the whole production chain and, thus, the existing inefficiency.

The exact mechanism is the following. Assume the monitoring threshold in the upstream sector is reduced from $\bar{y}$ to $\bar{y}^{'} < \bar{y}$, so only smaller firms can remain informal now. Given equilibrium prices p_i and p_f, the smaller threshold increases the mass of formal firms and reduces informality in the upstream stage. This relative increase in the supply of formal goods will decrease p_f and increase p_i, which will shift the informality threshold $\bar{\theta}$ to the left in the downstream sector, increasing formal production of the final good. The intuition is that, given that now buying formal intermediate goods is relatively cheaper, the marginal informal firms in the downstream sector will find it more profitable to become formal under lower prices. In this way, formality in the dowsntream sector increases.

Analogously, assume that the monitoring threshold in the downstream sector is reduced to $\bar{y}_d^{'} < \bar{y}_d$. This increases the mass of firms that operate formally in the downstream stage, increasing the demand for the formal intermediate good. This generates a relative increase in the formal price p_f, which increases profitability of formal firms in the upstream stage. Marginal upstream firms will find it profitable to become formal under these new prices.

Of course, the effect of attacking informality in the upstream or downstream sectors need not be equivalent, given that varying monitoring in the upstream shifts the supply of informal intermediate goods, whereas varying the threshold in the downstream shifts the demand. The final effect on relative prices depends strongly on the price elasticities of supply and demand for the intermediate goods.

2 Environment

The productive sector is composed by two types of firms: upstream firms, that produce an intermediate good used in the production of the final good, and downstream firms, that produce the final good. There is a continuum of entrepreneurs in each stage of production.

2.1 The Upstream Sector

The entrepreneurs that produce in the upstream sector differ on their managerial ability $\theta_u > 0$, where the distribution of abilities is given by $g_u(\cdot)$. A firm in the upstream sector with a manager with ability θ_u uses factors of production n_u, such as labor, and produces the intermediate good y according to the production function f, such that $y_u = f_u(\theta_u, n_u)$. If the firm employs n_u units of the factors of production, it has to pay a cost $c_u(\cdot)$, where c_u is a convex function. Firms in the upstream sector can choose to be formal and pay value-added tax (VAT) on their activity, or produce in the informal sector and avoid paying taxes, subject to the probability of being observed by the tax enforcement authorities and losing all profits. For simplicity, we assume that firms that produce below a threshold level $\bar{y}_u$ have a zero probability of being detected by the tax authorities, while those producing above $\bar{y}_u$ are detected with probability equal to one. In equilibrium, some firms will choose to produce in the informal sector, given the managerial ability of its entrepreneur. The profits of a formal firm in the upstream sector are:

$$\Pi_f^u(\theta_u, p_f) = \max_{n_u} \quad (1-\tau)p_f f_u(\theta_u, n_u) - c_u(n_u)$$

where τ is the VAT, p_f is the price of the upstream good produced by the formal firm, and θ_u is the ability of the firm's manager. Solving the formal upstream firm's problem yields a supply function for formal intermediate goods $y^f(\theta_u, p_f)$ given by:

$$y^f(\theta_u, p_f) = (1-\tau)p_f \frac{\partial f_u(\theta_u, n_u)}{\partial n_u}$$

One can see that $\partial y^f / \partial p_f > 0$ and, assuming complementarity between inputs and managerial ability of the manager[1] we have that $\partial y^f / \partial \theta_u > 0$, so $y^f(\theta_u, p_f)$ is strictly increasing in θ_u and p_f. Similarly, the profits of an informal firm are:

$$\Pi_i^u(\theta_u, p_i) = \max_{n_u} \quad p_i f_u(\theta_u, n_u) - c_u(n_u), \quad \text{subject to: } f_u(\theta_u, n_u) \leq \bar{y}_u$$

where p_i is the price of the informal upstream good and the firm faces a constraint on the output produced, such that the maximum amount it can produce while being informal is exactly $\bar{y}_u$. Given that the unconstrained supply function is strictly increasing, there exists a threshold $\underline{\theta}_u$ such that firms with $\theta_u \leq \underline{\theta}_u$ are not constrained, whereas informal firms with $\theta_u > \underline{\theta}_u$ produce exactly $y_u^i(\theta_u, n_u) = \bar{y}_u$. Namely, $\underline{\theta}_u$ is implicitly defined by $p_i \frac{\partial f_u(\underline{\theta}_u, p_i)}{\partial n_u} = \bar{y}_u$. Therefore, the supply of informal firms is weakly increasing in θ_u and p_i, and is given by:

$$y_u^i(\theta_u, p_i) = \begin{cases} p_i \frac{\partial f_u(\theta_u, n_u)}{\partial n_u}, & \text{for} \quad \theta_u \leq \underline{\theta}_u \\ \bar{y}_u \quad , & \text{for } \theta_u > \underline{\theta}_u \end{cases}$$

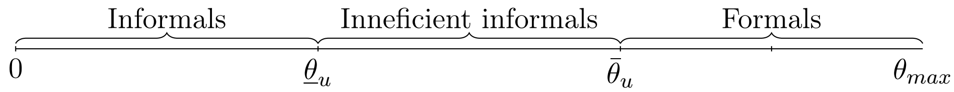
In equilibrium, for there to be a positive mass of firms producing in the informal sector, it must be the case that p_i and p_f are such that $\Pi_i^u(\theta_u, p_i) \geq \Pi_f^u(\theta_u, p_f)$ for some positive measure of abilities θ_u. If prices are such, there exists a unique threshold $\bar{\theta}_u$, implicitly defined by the equality $\Pi_i^u(\bar{\theta}_u, p_i) = \Pi_f^u(\bar{\theta}_u, p_f)$, such that firms with $\theta_u \geq \bar{\theta}_u$ operate in the formal sector, while firms with $\theta_u < \bar{\theta}_u$ are informal[2]. Moreover, if an informal sector exists, there is a positive mass of firms

[1] That is, assuming that $\frac{\partial^2 f_u(\theta_u, n_u)}{\partial n_u \partial \theta_u} > 0$.

[2] Take, for example, $(1-\tau)p_f \leq p_i$. If $\bar{y}_u > 0$, for a sufficiently low $\theta_u \leq \underline{\theta}_u$ it is straightforward to conclude that $\Pi_i^u(\theta_u, p_i) \geq \Pi_f^u(\theta_u, p_f)$. If we assume $\frac{\partial f_u(\theta_u, n_u)}{\partial \theta_u \partial n_u} > 0$ and $\frac{\partial^2 f_u(\theta_u, n_u)}{\partial \theta_u^2} < 0$, then the optimal unconstrained labor demand $n_u(\theta_u, p)$ for $\theta_u \leq \underline{\theta}_u$ is increasing in θ_u and p. For $\theta_u > \underline{\theta}_u$, instead, the constrained labor demand $n_u^c(\theta_u)$ is such that $\frac{\partial n_u}{\partial \theta_u} = -\frac{\partial f_u(\theta_u, n_u)/\partial \theta_u}{\partial f_u(\theta_u, n_u)/\partial n_u} < 0$. Using the envelope theorem:

$$\frac{\partial \Pi_i^u(\theta_u, p_i)}{\partial \theta_u} = \begin{cases} p_i \frac{\partial f_u(\theta_u, n_u(\theta_u, p_i))}{\partial \theta_u}, & \text{if } \theta_u \leq \underline{\theta}_u \\ p_i \frac{\partial f_u(\theta_u, n_u^c(\theta_u))}{\partial \theta_u}, & \text{if } \theta_u > \underline{\theta}_u \end{cases} \quad , \quad \frac{\partial \Pi_f^u(\theta_u, p_f)}{\partial \theta_u} = (1-\tau)p_f \frac{\partial f_u(\theta_u, n_u(\theta_u, p_f))}{\partial \theta_u}$$

$\theta_u \in [\underline{\theta}_u, \bar{\theta}_u]$ that will choose to produce in the informal sector and constrain their optimal size, which represents an inefficiency in the economy (Garicano et al. 2016).



The total supply of upstream informal and formal goods is determined, respectively, by:

$$Y_u^i(p_i, p_f, \bar{y}) = \int_0^{\underline{\theta}_u} y_u^i(\theta, p_i) \cdot g_u(\theta) d\theta + \int_{\underline{\theta}_u}^{\bar{\theta}_u} \bar{y}_u \cdot g_u(\theta) d\theta \tag{1}$$

$$Y^f(p_i, p_f, \bar{y}) = \int_{\bar{\theta}_u}^{\theta_u^{max}} y_u^f(\theta, p_f) \cdot g_u(\theta) d\theta \tag{2}$$

Note that total supply of the formal and informal intermediate goods depend on both prices p_i and p_f, and on the detection threshold $\bar{y}$, given that the informality cutoff $\bar{\theta}_u$ is an implicit function of these variables. If constrained firms were to produce in the formal economy, they would produce $y^f(\theta, pf)$. However, to avoid being detected, they constrain their production to $\bar{y}_u$. This generates a loss in the output of the upstream sector equal to:

$$L_u(p_i, p_f, \bar{y}) = \int_{\underline{\theta}_u}^{\bar{\theta}_u} y_u^f(\theta, p_f) g_u(\theta) d\theta - \bar{y}_u \left[G_u(\bar{\theta}_u) - G_u(\underline{\theta}_u) \right] \tag{3}$$

This output loss represents a dead-weight loss in the economy. In the absence of the monitoring thresholds, informal firms would optimally produce more. However, given that informal firms want to avoid paying VAT, they produce less than the efficient level.

2.2 The Downstream Sector

The second type of firms produce the downstream good, using as an input for production the managerial ability of the entrepreneur θ_d, distributed according to $g_d(\cdot)$, the upstream good x, and labor n_d, such that the total output is $f_d(\theta_d, x, n_d)$. Firms in the downstream sector also choose whether to be formal or not, with the same consequences as firms in the upstream sector: firms that produce above a threshold $\bar{y}_d$ are detected by tax authorities with certain probability, while those that produce below $\bar{y}_d$ remain undetected. Moreover, there is no restriction regarding the

So $\frac{\partial \Pi_i^u(\theta_u, p_i)}{\partial \theta_u} > \frac{\partial \Pi_f^u(\theta_u, p_f)}{\partial \theta_u}$ for $\theta_u \leq \underline{\theta}_u$, but $\frac{\partial \Pi_i^u(\theta_u, p_i)}{\partial \theta_u} < \frac{\partial \Pi_f^u(\theta_u, p_f)}{\partial \theta_u}$ for $\theta_u > \underline{\theta}_u$. Assuming f_u and C_u are such that Π is continuous, this implies the existence of $\bar{\theta}_u$.

source of the inputs; firms can buy their inputs in the formal or informal upstream markets. The only difference relies on the deductibility of the VAT: formal firms that buy their input in informal markets do not receive the tax credit on the VAT paid, whereas those that buy in formal markets can deduct the VAT. The profits of a formal downstream firm are:

$$\Pi_f^d(\theta_d, p_f) = \max\{\underbrace{\max_{n_d,x} \quad (1-\tau)\left(f_d(\theta_d,x,n_d) - p_f x\right) - c_d(n_d)}_{\text{Formal inputs}}, \quad \underbrace{\max_{n_d,x} \quad (1-\tau) f_d(\theta_d,x,n_d) - p_i x - c_d(n_d)}_{\text{Informal inputs}}\}$$

In the case in which $p_i \geq (1-\tau)p_f$, producers of the downstream good in the formal sector will weakly prefer to buy from formal producers, which will allow them to receive a credit for the VAT. In the equilibrium of a reasonable calibration of the model this will be the case, so we will omit the case in which formal firms choose to buy their inputs in the upstream informal markets. Solving the formal downstream firm's problem yields a demand function for the intermediate good $x_d^f(\theta_d, p_f)$ which is increasing in θ_d and decreasing in p_f.[3]

In the informal sector, downstream firms can also choose whether to buy the intermediate good from formal or informal firms. However, given that informal firms do not pay VAT, they cannot claim the tax credit, so the profits are:

$$\begin{aligned}\Pi_i^d(\theta_d, p_f) \quad &= \quad \max\{\underbrace{\max_{n_d,x} \quad f_d(\theta_d,x,n_d) - p_f x - c_d(n_d)}_{\text{Formal inputs}}, \quad \underbrace{\max_{n_d,x} \quad f_d(\theta_d,x,n_d) - p_i x - c_d(n_d)}_{\text{Informal inputs}}\} \\ & \qquad \text{subject to: } f_d(\theta_d,x,n_d) \leq \bar{y}_d\end{aligned}$$

Note that informal firms will buy only from informal upstream producers so long as $p_i \leq p_f$, and only from formal producers otherwise. Again, this condition is satisfied in equilibrium for there to exist an informal sector, so we will omit the case in which informal firms choose to buy from formal upstream firms. Therefore, for there to exist informal upstream producers, we will assume henceforth that, in equilibrium, $(1-\tau)p_f \leq p_i \leq p_f$, and informal downstream producers buy only intermediate goods from informal upstream producers. Solving the downstream informal firm's

[3] Assuming complementarity between inputs and managerial ability, $\frac{\partial^2 f_d(\theta_d,x,n_d)}{\partial x \partial \theta_d} > 0$, and decreasing marginal returns of the intermediate good, $\frac{\partial^2 f_d(\theta_d,x)}{\partial x^2} < 0$, the implicit function theorem yields the result:

$$\frac{\partial x_d^f(\theta_d,p_f)}{\partial \theta_d} = -\frac{\partial^2 f_d(\theta_d,x,n_d)/\partial x \partial \theta_d}{\partial^2 f_d(\theta_d,x,n_d)/\partial x^2} > 0, \qquad \frac{\partial x_d^f(\theta_d,p_f,n_d)}{\partial p_f} = \frac{1}{\partial^2 f_d(\theta_d,x,n_d)/\partial x^2} < 0$$

problem yields a demand function for the intermediate good $x^i(\theta_d, p_i)$. As in the upstream sector, it can be shown that there exists a threshold $\underline{\theta}_d$ such that the output constraint $\bar{y}_d$ does not bind for downstream informal firms with $\theta_d \leq \underline{\theta}_d$, so solving the downstream informal firm's problem for $\theta_d \leq \underline{\theta}_d$ yields a demand for the intermediate good $x^i(\theta_d, p_i)$ that is increasing in θ_d and decreasing in p_i. In contrast, the constraint binds for informal firms with entrepreneurial ability $\theta_d > \underline{\theta}_d$, so the firm's problem yields a demand for the intermediate good $x^i_c(\theta_d, p_i)$ that is decreasing in θ_d and p_i.

The downstream firms choose to be formal whenever $\Pi^d_f(\theta_d, p_f) \geq \Pi^d_i(\theta_d, p_i)$. As in the upstream sector, there exists a cut-off ability $\bar{\theta}_d$, defined implicitly by the equality $\Pi^d_f(\bar{\theta}_d, p_f) = \Pi^d_i(\bar{\theta}_d, p_i)$, such that entrepreneurs with $\theta_d < \bar{\theta}_d$ choose to be informal, whereas those with $\theta_d \geq \bar{\theta}_d$ are formal. The left panel in Figure 1 illustrates the profits and input demand of formal and informal buyers, where the dashed line is the value $\bar{\theta}_d$ at which the entrepreneur is indifferent between being formal or informal. Below the dashed line, entrepreneurs prefer to produce in the informal sector and be potentially constrained. Only those with ability above the dashed line prefer to be formal. Firms

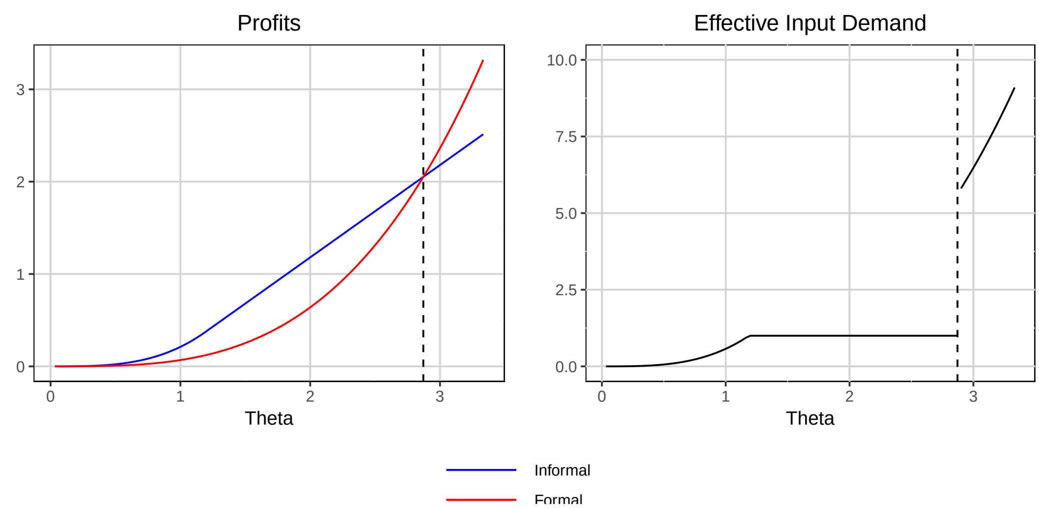


Figure 1: Profits and input demand of formal and informal firms.

with $\theta_d \in [\underline{\theta}_d, \bar{\theta}_d]$ will inefficiently produce exactly $\bar{y}_d$ to avoid being detected. The right panel in Figure 1 illustrates the effective output of the final good when there is VAT evasion in the economy. Those firms with $\theta_d < \bar{\theta}_d$ choose an input demand that allows them to remain non-monitored by the tax authorities, so they effectively pay no VAT. However, those above the dashed line have sufficiently large profits in the formal sector, so they choose to pay VAT. Therefore, the output of the final good and the input demand $x^i(\theta_d, p_i)$ have a discontinuity in $\bar{\theta}_d$, given that formal producers do not face the output constraint. The total demand for the intermediate upstream good in the informal and formal downstream sectors are, respectively:

$$X^i(p_i, p_f, \bar{y}_d) = \int_0^{\underline{\theta}_d} x^i(\theta, p_i) \cdot g_d(\theta) d\theta + \int_{\underline{\theta}_d}^{\bar{\theta}_d} x_c^i(\theta, p_i) \cdot g_d(\theta) d\theta \tag{4}$$

$$X^f(p_i, p_f, \bar{y}_d) = \int_{\bar{\theta}_d}^{\theta_d^{max}} x^f(\theta, p_f) \cdot g_d(\theta) d\theta \tag{5}$$

The total informal and formal production in the downstream sector is given by:

$$Y_d^i(p_i, p_f, \bar{y}_d) = \int_0^{\underline{\theta}_d} y_d^i(\theta, p_i) \cdot g_d(\theta) d\theta + \int_{\underline{\theta}_d}^{\bar{\theta}_d} y_{d,c}^i(\theta, p_i) \cdot g_d(\theta) d\theta \tag{6}$$

$$Y_d^f(p_i, p_f, \bar{y}_d) = \int_{\bar{\theta}_d}^{\theta_d^{max}} y^f(\theta, p_f) \cdot g_d(\theta) d\theta \tag{7}$$

where $y(\cdot)$ denotes the optimal supply in each case (informal, informal-constrained, and formal). Note that in the downstream sector we also observe a loss in production as constrained firms would produce a different level if they were to produce according to their optimal formal supply. This loss is defined as:

$$L_d(p_i, p_f, \bar{y}_d) = \int_{\underline{\theta}_d}^{\bar{\theta}_d} y^f(\theta, p_f) g_d(\theta) d\theta - \bar{y} \left[G_d(\bar{\theta}_d) - G_d(\underline{\theta}_d) \right] \tag{8}$$

2.3 How Do the Detection Thresholds Affect Welfare?

To understand the welfare effects of an increase in government monitoring, we must first understand the inefficiencies that exist in the economy. Recall that, given the presence of taxation and imperfect monitoring, there is a mass of firms in the informal sector for whom the output constraint is binding. These firms inefficiently produce $\bar{y}_u$ and $\bar{y}_d$ in the upstream and downstream sectors, respectively, instead of producing their optimal unconstrained amount. Equations (3) and (8) characterize the output loss of constrained informal firms in each sector. The dead-weight loss generated by the inefficiently low output of informal firms is directly proportional to the size of the informal sector in both stages of production; a larger informal sector represents a larger loss of output of the informal firms that want to avoid monitoring. Thus, to understand the welfare effects of varying the monitoring thresholds $\bar{y}_u$ and $\bar{y}_d$, we must understand how informality responds to these changes.

The size of informality in the upstream and downstream sectors depends on the values of the thresholds $\bar{\theta}_u$ and $\bar{\theta}_d$. These thresholds, in turn, depend on the detection thresholds $\bar{y}_u$ and $\bar{y}_d$ and on the prices p_i and p_f. Therefore, in order to assess the effect of varying the monitoring thresholds $\bar{y}_u$ and $\bar{y}_d$ on informality, we first have to assess the impact of varying these thresholds on p_i and p_f.

In equilibrium, there is market clearing in the formal and informal input markets. This means that, given detection thresholds $\bar{y}_u$ and $\bar{y}_d$, the prices p_f and p_i for the formal and informal intermediate goods are implicitly defined by the market clearing conditions:

$$X^i(p_i, p_f, \bar{y}_d) = Y^i(p_i, p_f, \bar{y}_u) \tag{9}$$

$$X^f(p_i, p_f, \bar{y}_d) = Y^f(p_i, p_f, \bar{y}_u) \tag{10}$$

Moreover, given prices p_f and p_i, the informality thresholds $\bar{\theta}_u$ and $\bar{\theta}_d$ in the upstream and downstream sectors are implicitly defined by the indifference conditions:

$$\Pi_i^u(\bar{\theta}_u, p_i) = \Pi_f^u(\bar{\theta}_u, p_f) \tag{11}$$

$$\Pi_f^d(\bar{\theta}_d, p_f) = \Pi_i^d(\bar{\theta}_d, p_i) \tag{12}$$

The following proposition characterizes the relation between the detection thresholds $\bar{y}_u$ and $\bar{y}_d$ and the equilibrium prices:

Proposition 1. *If* $\frac{x^f(\bar{\theta}_d, p_f)}{\bar{x}} > \frac{y^f(\bar{\theta}_u, p_f)}{\bar{y}}$, *the derivative* $\frac{\partial p_i}{\partial \bar{y}} < 0$. *The signs of* $\frac{\partial p_f}{\bar{y}}$, $\frac{\partial p_i}{\bar{x}}$, *and* $\frac{\partial p_f}{\bar{y}_d}$ *are undetermined.*

Proof. The proof is on the Appendix. □

Crucially, the signs of the derivatives of prices with respect to the detection thresholds are undetermined, except for $\frac{\partial p_i}{\partial \bar{y}}$. These signs depend, among other things, on the size of the formal and informal sectors, on the size of the constrained firms within the informal sector, on the average price elasticities of demand and supply for the intermediate upstream good among formal and informal firms, and on the distribution of firms. The next proposition states that it is unclear whether...

As shown in the appendix, the informality thresholds, implicitly defined by equations (11) and (12), also depend ambiguously on the $\bar{y}$ and $\bar{x}$. Moreover, the impact of varying $\bar{y}$ and $\bar{x}$ on informality in the upstream and downstream sectors depends.

3 Quantitative Model

In this section, we describe the quantitative model. The production side of the economy is analogous to the one described in the past section. We add a household side to close the economy and study the general equilibrium effects of different monitoring policies. However, the representative household in our model will play a passive role as, we think, most of the action occurs in the production side of the economy.

3.1 Production

This section describes the functional forms adopted for the production and firm distributions to calibrate the model to the Mexican economy. Particularly, we assume that the managerial ability is distributed on both sectors with a Pareto distribution, whose parameters are calibrated to match the observed distributions.

3.1.1 Upstream Sector

Firms in the upstream sector use labor l and entrepreneurial ability θ_u to produce the intermediate good according to a Cobb-Douglas technology:

$$f_u(l, \theta_u) = \theta_u l^{\alpha_u} \tag{13}$$

Labor is hired in a competitive market at a wage w per unit. Formal firms pay VAT taxes at a rate τ on their value added. Informal firms do not pay taxes. However, informal firms producing beyond a given threshold $\bar{y}_d$ will be detected by the tax authority with probability one. For such a reason, an informal firm should produce less than $\bar{y}_d$. Formal firms sell their product at a price p_f and informal firms do so at a price p_i. Profits for formal firms in the upstream sector are described by:

$$\Pi_u^f(\theta_u, p_f, w) = \max_{l} \quad (1-\tau)p_f\theta_u l^{\alpha_u} - wl \tag{14}$$

Informal firms in the upstream sector have profits described by:

$$\Pi_u^i(\theta_u, p_i, w) = \max_{l \leq \bar{l}(\bar{y}_u, \theta_u)} p_i\theta_u l^{\alpha_u} - wl \tag{15}$$

where $\bar{l}(\bar{y}_u, \theta_u)$ is the amount of labor required to produce $\bar{y}_u$. An informal firm will be constrained by size up to a level $\bar{y}_u$ beyond which she will be observed by the tax authorities. Note that for there to exist an informal sector it should be the case that $p_i \geq (1-\tau)p_f$.

An informal firm in the upstream sector will be constrained if the value of entrepreneurial ability is such that, in the absence of the production constraint in the informal sector $\bar{y}_u$, such a firm would choose to produce above $\bar{y}_u$. Firms with productivity levels $\theta_u > \underline{\theta}_u(\bar{y}_u, w, p_i)$ that are part of the informal sector are constrained, where $\underline{\theta}_u(\bar{y}_u, w, p_i)$ is defined as:

$$\underline{\theta}_u(\bar{y}_u, w, p_i) := \bar{y}_u^{1-\alpha_u} \left(\frac{w}{\alpha_u p_i} \right)^{\alpha_u} \tag{16}$$

An upstream firm with productivity levels higher than $\underline{\theta}_u(\bar{y}_u, w, p_i)$ will decide to produce in the informal sector so long as its profits exceed those in the formal sector. The decision will be then to produce in the formal sector whenever $\Pi_u^f(\theta_u, p_f, w) > \Pi_u^i(\theta_u, p_i, w)$ or, equivalently, when:

$$\left(\frac{(1-\tau)p_f\theta_u}{w^{\alpha_u}} \right)^{\frac{1}{1-\alpha_u}} \left(\alpha_u^{\frac{\alpha_u}{1-\alpha_u}} - \alpha_u^{\frac{1}{1-\alpha_u}} \right) \geq p_i\bar{y}_u - w \left(\frac{\bar{y}_u}{\theta_u} \right)^{\frac{1}{\alpha_u}} \tag{17}$$

We define $\hat{\theta}_u(\bar{y}_u, p_i, p_f, w, \tau)$ the productivity level at which an entrepreneur is indifferent between being informal and constrained, or being formal. In other words, $\hat{\theta}_u(\bar{y}_u, p_i, p_f, w, \tau)$ makes Equation (17) hold with equality. Entrepreneurs for which $\theta > \hat{\theta}_u(\bar{y}_u, p_i, p_f, w, \tau)$ will produce in the formal sector.

In order to match the firm distribution, we will assume that the entrepreneurial ability θ_u is distributed according to a Pareto distribution with parameters $(\gamma_u, \theta_{u,\min})$. Under this assumption, the supply of the informal and formal upstream goods are, respectively:

$$\begin{aligned} Y_u^i(p_i, p_f, w, \tau, \bar{y}_u) &= (\theta_u^{\min})^{\gamma_u} \left[\left(\frac{\alpha_u p_i}{w} \right)^{\frac{\alpha_u}{1-\alpha_u}} \left(\frac{\gamma_u(1-\alpha_u)}{\gamma_u(1-\alpha_u)-1} \right) \left((\theta_u^{\min})^{\frac{1+\alpha_u\gamma_u-\gamma_u}{1-\alpha_u}} - \underline{\theta}_u(\bar{y}_u)^{\frac{1+\alpha_u\gamma_u-\gamma_u}{1-\alpha_u}} \right) \right. \\ &+ \left. \bar{y}_u \left[\left(\frac{1}{\underline{\theta}_u(p_i, p_f, w, \bar{y}_u, \tau)} \right)^{\gamma_u} - \left(\frac{1}{\hat{\theta}_u(p_i, p_f, w, \bar{y}_u, \tau)} \right)^{\gamma_u} \right] \right] \end{aligned} \tag{18}$$

$$Y_u^f(p_i, p_f, w, \tau, \bar{y}_u) = \left(\frac{(1-\tau)\alpha_u p_f}{w} \right)^{\frac{\alpha_u}{1-\alpha_u}} \left(\frac{\gamma_u(1-\alpha_u)}{\gamma_u(1-\alpha_u)-1} \right) \frac{(\theta_u^{\min})^{\gamma_u}}{\hat{\theta}_u(p_i, p_f, \bar{y}, w, \tau)^{\gamma_u(1-\alpha)-1}} \tag{19}$$

so long as $\gamma_u > \frac{1}{1-\alpha_u}$.

JS: En el denominador es $\hat{\theta}_u(p_i, p_f, \bar{y}, w, \tau)^{\frac{\gamma_u(1-\alpha_u)-1}{1-\alpha_u}}$

Note that, given the functional assumptions made so far, we can obtain closed-form solutions to the output produced formally and informally in the upstream sector, except for the threshold level $\hat{\theta}_u$, which is implicitly defined by the equality in equation (17).

3.1.2 Downstream Sector

Firms in the downstream sector produce according to their productivity level θ_d, and use labor and production from the upstream sector to generate output according to the technology:

$$f_d(\theta_d, x, l) = \theta_d l^{\alpha_d} x^{\beta_d} \tag{20}$$

where $\alpha_d + \beta_d < 1$. Downstream formal firms pay corporate income taxes at a rate τ. If they use inputs from the formal sector, they can deduct the costs by claiming VAT credit. As informal firms do not pay taxes, they will not demand from the formal sector unless its price is lower than that of the informal one. For this reason, only formal firms in the downstream sector will demand formal goods form the upstream sector. When deciding weather to buy from the formal or the informal sector, a downstream formal firm is faced with the following problem:

$$\Pi_d^f(\theta_d, p_f, w) = \max\{\underbrace{\max_{l,x} \quad (1-\tau)\left(\theta_d l^{\alpha_d} x^{\beta_d} - p_f x\right) - wl,}_{\text{Buy inputs in formal markets}} \quad \underbrace{\max_{l,x} \quad (1-\tau)\theta_d l^{\alpha_d} x^{\beta_d} - p_i x - wl\}}_{\text{Buy inputs in informal markets}} \tag{21}$$

From Equation (21) it is clear that for formal firms to exist in the upstream sector, it needs to be the case that $p_f(1-\tau) < p_i$. Solving for the formal downstream firm's problem, profits are:

$$\Pi_d^f(\theta_d, p_f, w) = (1-\tau)(1-\alpha_d-\beta_d)\left[\theta_d(1-\tau)^{\alpha_d}\left(\frac{\alpha_d}{w}\right)^{\alpha_d}\left(\frac{\beta_d}{p_f}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} \tag{22}$$

As in the upstream sector, informal firms are not be able to produce beyond a point $\bar{y}_d$, above which they will be detected by the tax authorities. The informal downstream firm's problem is:

$$\Pi_d^i(\theta_d, p_i, w) = \max\{\underbrace{\max_{l,x} \quad \theta_d l^{\alpha_d} x^{\beta_d} - p_f x - wl,}_{\text{Buy inputs in formal markets}} \quad \underbrace{\max_{l,x} \quad \theta_d l^{\alpha_d} x^{\beta_d} - p_i x - wl\}}_{\text{Buy inputs in informal markets}} \tag{23}$$

$$\text{s.t.:} \quad \theta_d l^{\alpha_d} x^{\beta_d} \leq \bar{y}_d \tag{24}$$

The profits of informal firms that are constrained and choose to produce exactly $\bar{y}_d$ to avoid being detected are thus:

$$\Pi^i_{d,c}(\bar{y}_d, w, p_i, \theta_d) = \bar{y}_d - \left(\frac{\bar{y}_d}{\theta_d}\right)^{\frac{1}{\alpha_d+\beta_d}} \left[\left(w^{\alpha_d} p_i^{\beta_d}\right)^{\frac{1}{\alpha_d+\beta_d}} \left[\left(\frac{\alpha_d}{\beta_d}\right)^{\frac{\beta_d}{\alpha_d+\beta_d}} + \left(\frac{\beta_d}{\alpha_d}\right)^{\frac{\alpha_d}{\alpha_d+\beta_d}}\right]\right] \tag{25}$$

Similarly, the profits of informal firms whose production is not constrained by the threshold $\bar{y}_d$ are:

$$\Pi^i_{d,u}(\theta_d, w, p_f) = (1 - \alpha_d - \beta_d) \left[\theta_d \left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_i}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} \tag{26}$$

As already stated, there are cut-offs that define the firms that are formal and informal and, within informal firms, those that are constrained and unconstrained. The value $\underline{\theta}_d$ above which firms will choose to optimally produce $\bar{y}_d$ to avoid being detected is:

$$\underline{\theta}_d(\bar{y}, w, p_i) = \bar{y}_d^{1-\alpha_d-\beta_d} \left(\frac{w}{\alpha_d}\right)^{\alpha_d} \left(\frac{p_i}{\beta_d}\right)^{\beta_d} \tag{27}$$

Finally, a downstream firm will decide to be informal or formal by comparing the corresponding profits. This decision is given by a threshold rule such that firms for which $\theta \geq \hat{\theta}_d(p_i, p_f, w, \bar{y})$ will decide to be formal. The threshold $\theta \geq \hat{\theta}_d(p_i, p_f, w, \bar{y})$ is defined implicitly by the ability for which profits in the formal and informal sectors are equal:

$$\bar{y}_d - \left(\frac{\bar{y}_d}{\hat{\theta}_d(p_i, p_f, w, \bar{y})}\right)^{\frac{1}{\alpha_d+\beta_d}} \left[\left(w^{\alpha_d} p_i^{\beta_d}\right)^{\frac{1}{\alpha_d+\beta_d}} \left[\left(\frac{\alpha_d}{\beta_d}\right)^{\frac{\beta_d}{\alpha_d+\beta_d}} + \left(\frac{\beta_d}{\alpha_d}\right)^{\frac{\alpha_d}{\alpha_d+\beta_d}}\right]\right] =$$

$$(1-\tau)(1 - \alpha_d - \beta_d) \left[\hat{\theta}_d(p_i, p_f, w, \bar{y})(1-\tau)^{\alpha_d} \left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_i}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} \tag{28}$$

If we assume that entrepreneurial ability of firms in the downstream sector are distributed to a Pareto distribution with parameters $(\gamma_d, \theta_{d,min})$, the corresponding unconstrained informal demand is given by:

$$X_d^{i,u}(p_i, \bar{y}_d, w) = \left[\left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_i}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} \left(\frac{\gamma_d}{\kappa_d + 1}\right) \left(\theta_d^{\min}\right)^{\gamma_d} \left(\underline{\theta}_d^{\kappa_d+1} - \left(\theta_d^{\min}\right)^{\kappa_d+1}\right) \tag{29}$$

JS: Aquí el exponente es $1 - \alpha_d$

with κ_d defined as follows:

$$\kappa_d = \frac{\alpha_d + \beta_d - \gamma_d(1 - \alpha_d - \beta_d)}{1 - \alpha_d - \beta_d} \tag{30}$$

The informal constrained demand is:

$$X_d^{i,c}(p_i, p_f, \bar{y}_d, w, \tau) = \left[\bar{y}_d \left(\frac{w\beta_d}{p_i\alpha_d}\right)^{\alpha_d}\right]^{\frac{1}{\alpha_d+\beta_d}} \left(\frac{\gamma_d}{c_d}\right) (\theta_d^{\min})^{\gamma_d} \left(\hat{\theta}(p_i, p_f, w, \bar{y}, \tau)^{c_d} - \underline{\theta}_d^{c_d}\right) \tag{31}$$

with

$$c_d = \frac{1 - \gamma_d(\alpha_d + \beta_d)}{\alpha_d + \beta_d} \tag{32}$$

JS:

$$c_d = -\frac{1 + \gamma_d(\alpha_d + \beta_d)}{\alpha_d + \beta_d} \tag{33}$$

Under the assumption that $\alpha_d + \beta_d < 1$ and $\gamma_d > 1$, the input demand of formal downstream firms is given by:

$$X_d^f(p_i, p_f, \bar{y}_d, w, \tau) = \left[\left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_f}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} \left(\frac{\gamma_d}{\kappa_d}\right) (\theta_d^{\min})^{\gamma_d} \left(-\left(\hat{\theta}_d(p_i, p_f, w, \bar{y}_d, \tau)\right)^{\kappa_d}\right) \tag{34}$$

JS:

$$X_d^f(p_i, p_f, \bar{y}_d, w, \tau) = \left[(1-\tau)^{\alpha_d} \left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_f}\right)^{1-\alpha_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} \left(\frac{\gamma_d}{\kappa_d + 1}\right) (\theta_d^{\min})^{\gamma_d} \left(-\left(\hat{\theta}_d(p_i, p_f, w, \bar{y}_d, \tau)\right)^{\kappa_d+1}\right)$$

with condition being $\gamma_d > \frac{1}{1-\alpha_d-\beta_d}$

As for the upstream sector, we obtain almost-closed-form expressions for the input demands, except for the threshold $\hat{\theta}_d$, which is implicitly defined by equation (28).

3.2 Households

The economy is composed of a representative household, which has preferences for formal and informal goods. There may be reasons to think that formal and informal goods are not perfectly substitutable. For instance, formal goods usually come with a warranty that allows the buyer to change or return the item if the quality of the item is not as initially expected, they can be purchased in nicer and cleaner markets, and users might feel satisfaction of paying taxes. On the other hand, informal goods are cheaper, buying in informal markets usually allows buyers to bargain and/or receive credit by the buyer, and informal markets are usually smaller stores that are spread across cities. Therefore, we assume that households have a utility $u(c_f, c_i)$ that depends on consumption of the formal c_f and informal c_i goods, where the goods may not be perfectly substitutable:

$$u(c_f, c_i) = \frac{(\gamma c_f^{\phi} + (1-\gamma) c_i^{\phi})^{\frac{1-\sigma}{\phi}}}{1-\sigma}$$

where γ is the relative weight of formal goods in the utility function, $1/\phi$ is the elasticity of substitution between formal and informal goods, and σ is the coefficient of risk aversion. In this specification, if $\phi = 1$, the formal and informal goods are identical and perfectly substitutable.

Households are endowed with a unit of time, which they inelastically supply as labor to firms in exchange for a wage w, determined in competitive markets. In addition to labor income, households own the upstream and downstream firms, so every period they obtain the profits from both sectors.

The recursive formulation of the household's problem is:

$$V(a) = \max_{c_i, c_f} \frac{(\gamma c_f^{\phi} + (1-\gamma) c_i^{\phi})^{\frac{1-\sigma}{\phi}}}{1-\sigma} + \beta V(a') \tag{35}$$

$$(1+\tau)c_f + q_i c_i + a' = wn + (1+r)a + \Pi_u + \Pi_d$$

where q_i is the price of the final good sold in informal markets relative to the good sold in formal markets, Π_u are the profits of all the firms (formal and informal) in the upstream sector, and Π_d are the profits of all the firms in the downstream sector.

3.3 Equilibrium

A competitive equilibrium in this environment are prices p_i, p_f, a wage w, labor demand functions $l_f(\theta, p_i, p_f, w), l_{i,u}(\theta, p_i, p_f, w)$ and $l_{i,c}(\theta, p_i, p_f, w)$, cut-off functions $\hat{\theta}_u, \hat{\theta}_d, \underline{\theta}_u$, and $\underline{\theta}_d$, policy functions for households $c(a), a'(a)$, such that:

1. Firms optimize:

   (a) Firms with $\theta_u > \hat{\theta}_u$ operate formally and choose labor input $l_f^u(\theta, p_i, p_f, w)$ to solve (14), while firms with $\theta_u \leq \hat{\theta}_u$ are part of the upstream informal sector and choose $l_i^u(\theta, p_i, p_f, w)$ to solve (15).

(b) Firms with $\theta_d > \hat{\theta}_d$ operate formally and choose labor $l_f^d(\theta, p_i, p_f, w)$ and inputs $x_f^d(\theta, p_i, p_f, w)$ to solve (21), while firms with $\theta_d \leq \hat{\theta}_d$ are part of the upstream informal sector and choose labor $l_i^d(\theta, p_i, p_f, w)$ and input $x_i^d(\theta, p_i, p_f, w)$ to solve (23).

2. Households choose consumption $c_i(a), c_f(a)$ and savings $a'(a)$ policy functions to solve (35).

3. Markets clear:

(a) Input markets:

$$X^i(p_i, p_f, \bar{y}_d) = Y_u^i(p_i, p_f, \bar{y}_u) \quad (36)$$

$$X^f(p_i, p_f, \bar{y}_d) = Y_u^f(p_i, p_f, \bar{y}_u) \quad (37)$$

(b) Labor market:

$$\int_0^{\underline{\theta}_u} l_{i,u}(\theta, p_i, w) dG_u(\theta) + \int_{\underline{\theta}_u}^{\hat{\theta}_u} l_{i,c}(\theta, p_i, w) dG_u(\theta) + \int_{\hat{\theta}_u}^{\theta_u^{max}} l_f(\theta, p_i, w) dG_u(\theta) = 1 \quad (38)$$

(c) Goods market:

$$Y_d^f(p_i, p_f, \bar{y}_u) = \int_{\mathcal{A}} c_f(a) da \quad (39)$$

$$Y_d^i(p_i, p_f, \bar{y}_u) = \int_{\mathcal{A}} c_i(a) da \quad (40)$$

4 Calibration

Table 1: Labor supply 2010

Parameter	Source
α_u	Workers' share(?)
α_d	Worker's share
β_d	Capital's share
$f_u(\theta_u)$	Pareto(?). Firm size distribution. One parameter to estimate.
$f_d(\theta_d)$	Pareto(?). Firm size distribution
$\bar{y}_u$	Informal share in upstream
$\bar{y}_d$	Informal share in downstream

To consider:

- w is a parameter in the partial equilibrium version of the model.
- Upstream-Downstream definition can be done in two ways. Pregunta de clientes más importantes o matriz insumo producto.

5 Results

6 Conclusion

Appendices

Let $y^i(\theta, p_i, \bar{y})$ be the optimal supply for the intermediate good in the informal sector, and $y^f(\theta, pf, \bar{y})$ in the formal sector.

In equilibrium, total demand and supply are given by:

$$y^i(p_i, p_f, \bar{y}) = \underbrace{\int_0^{\underline{\theta}^u} y^i(\theta, p_i, \bar{y}) \cdot g_u(\theta) d\theta}_{Unconstrained} + \underbrace{\int_{\underline{\theta}^u}^{\bar{\theta}^u} \bar{y}_u \cdot g_u(\theta) d\theta}_{Constrained}$$

$$y^f(p_i, p_f, \bar{y}) = \underbrace{\int_{\bar{\theta}^u}^{\theta^u_{max}} y^f(\theta, p_f, \bar{y}) \cdot g_u(\theta) d\theta}_{Unconstrained}$$

$$x^i(p_i, pf, \bar{x}) = \underbrace{\int_0^{\underline{\theta}^d} x^i(\theta, p_i, \bar{x}) \cdot g_d(\theta) d\theta}_{Unconstrained} + \underbrace{\int_{\underline{\theta}^d}^{\bar{\theta}^d} \bar{x} g_d(\theta) d\theta}_{Constrained}$$

$$x^f(p_i, pf, \bar{x}) = \underbrace{\int_{\bar{\theta}^d}^{\theta^d_{max}} x^f(\theta, pf, \bar{x}) \cdot g_d(\theta) d\theta}_{Unconstrained}$$

Using Leibniz rule, the derivatives that we will use below are:

$$
\begin{aligned}
\textcolor{red}{Chequeada}\frac{\partial y^i(p_i, pf, \bar{y})}{\partial p_i} &= y^i(\underline{\theta}^u, p_i, \bar{y}) \cdot g_u(\underline{\theta}^u)\frac{\partial \underline{\theta}^u}{\partial p_i} + \int_0^{\underline{\theta}^u} \frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i} \cdot g_u(\theta) d\theta + \\
&\quad \bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i} - \bar{y} g_u(\underline{\theta}^u)\frac{\partial \underline{\theta}^u}{\partial p_i} \\
&= \int_0^{\underline{\theta}^u} \frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i} \cdot g_u(\theta) d\theta + \bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i} \\
\textcolor{red}{Chequeada}\frac{\partial y^f(p_i, p_f, \bar{x})}{\partial p_f} &= -y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f} + \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f} \cdot g_u(\theta) d\theta \\
\textcolor{red}{Chequeada}\frac{\partial y^f(p_i, p_f, \bar{x})}{\partial p_i} &= -y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i} \\
\textcolor{red}{Chequeada}\frac{\partial y^i(p_i, pf, \bar{y})}{\partial \bar{y}} &= y^i(\underline{\theta}^u, p_i, \bar{y}) \cdot g_u(\underline{\theta}^u)\frac{\partial \underline{\theta}^u}{\partial \bar{y}} + \bar{y}\left(g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}} - g_u(\underline{\theta}^u)\frac{\partial \underline{\theta}^u}{\partial \bar{y}}\right) + \int_{\underline{\theta}^u}^{\bar{\theta}^u} g_u(\theta) d\theta \\
&= \bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}} + \int_{\underline{\theta}^u}^{\bar{\theta}^u} g_u(\theta) d\theta \\
\textcolor{red}{Chequeada}\frac{\partial y^i(p_i, p_f, \bar{x})}{\partial p_f} &= \bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f} \\
\textcolor{red}{Chequeada}\frac{\partial y^f(p_i, p_f, \bar{x})}{\partial \bar{y}} &= -y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}}
\end{aligned}
$$

$$\text{Chequeada}\frac{\partial x^i(p_i,p_f,\bar{x})}{\partial p_i} = x^i(\underline{\theta}^d,p_i,\bar{x})\cdot g_d(\underline{\theta}^d)\frac{\partial \underline{\theta}^d}{\partial p_i} + \int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta,p_i,\bar{x})}{\partial p_i}\cdot g_d(\theta)d\theta + \bar{x}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i} - \bar{x}g_d(\underline{\theta}^d)\frac{\partial \underline{\theta}^d}{\partial p_i}$$

$$= \int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta,p_i,\bar{x})}{\partial p_i}\cdot g_d(\theta)d\theta + \bar{x}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i}$$

$$\text{Chequeada}\frac{\partial x^i(p_i,p_f,\bar{x})}{\partial p_f} = \bar{x}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f}$$

$$\text{Chequeada}\frac{\partial x^i(p_i,p_f,\bar{x})}{\partial \bar{x}} = x^i(\underline{\theta}^d,p_i,\bar{x})\cdot g_d(\underline{\theta}^d)\frac{\partial \underline{\theta}^d}{\partial \bar{x}} + \int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta,p_i,\bar{x})}{\partial \bar{x}}\cdot g_d(\theta)d\theta + \bar{x}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}} - \bar{x}g_d(\underline{\theta}^d)\frac{\partial \underline{\theta}^d}{\partial \bar{x}} + \int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta$$

$$= \bar{x}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}} + \int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta$$

$$\text{Chequeada}\frac{\partial x^f(p_i,p_f,\bar{x})}{\partial p_f} = -x^f(\bar{\theta}^d,pf,\bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f} + \int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta,pf,\bar{x})}{\partial p_f}\cdot g_d(\theta)d\theta$$

$$\text{Chequeada}\frac{\partial x^f(p_i,p_f,\bar{x})}{\partial p_i} = -x^f(\bar{\theta}^d,pf,\bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i}$$

$$\text{Chequeada}\frac{\partial x^f(p_i,p_f,\bar{x})}{\partial \bar{x}} = -x^f(\bar{\theta}^d,pf,\bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}$$

Prices p_i and p_f are determined in equilibrium by the zero of the function G:

$$G(p_i,p_f,\bar{x},\bar{y}) = \begin{pmatrix} g_1(p_i,p_f,\bar{x},\bar{y}) \\ g_2(p_i,p_f,\bar{x},\bar{y}) \end{pmatrix} = \begin{pmatrix} x^i(p_i,p_f,\bar{x}) - y^i(p_i,p_f,\bar{y}) \\ x^f(p_i,p_f,\bar{x}) - y^f(p_i,p_f,\bar{y}) \end{pmatrix}$$

By the implicit function theorem:

$$
\begin{aligned}
\begin{pmatrix} \frac{\partial p_i}{\partial \bar{x}} & \frac{\partial p_i}{\partial \bar{y}} \\ \frac{\partial p_f}{\partial \bar{x}} & \frac{\partial p_f}{\partial \bar{y}} \end{pmatrix} &= -\begin{pmatrix} \frac{\partial g_1}{\partial p_i} & \frac{\partial g_1}{\partial p_f} \\ \frac{\partial g_2}{\partial p_i} & \frac{\partial g_2}{\partial p_f} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{\partial g_1}{\partial \bar{x}} & \frac{\partial g_1}{\partial \bar{y}} \\ \frac{\partial g_2}{\partial \bar{x}} & \frac{\partial g_2}{\partial \bar{y}} \end{pmatrix} \\
&= -\begin{pmatrix} \frac{\partial x^i}{\partial p_i} - \frac{\partial y^i}{\partial p_i} & \frac{\partial x^i}{\partial p_f} - \frac{\partial y^i}{\partial p_f} \\ \frac{\partial x^f}{\partial p_i} - \frac{\partial y^f}{\partial p_i} & \frac{\partial x^f}{\partial p_f} - \frac{\partial y^f}{\partial p_f} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{\partial x^i}{\partial \bar{x}} & -\frac{\partial y^i}{\partial \bar{y}} \\ \frac{\partial x^f}{\partial \bar{x}} & -\frac{\partial f^f}{\partial \bar{y}} \end{pmatrix} \\
&= A \begin{pmatrix} \frac{\partial x^f}{\partial p_f} - \frac{\partial y^f}{\partial p_f} & \frac{\partial y^i}{\partial p_f} - \frac{\partial x^i}{\partial p_f} \\ \frac{\partial y^f}{\partial p_i} - \frac{\partial x^f}{\partial p_i} & \frac{\partial x^i}{\partial p_i} - \frac{\partial y^i}{\partial p_i} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial x^i}{\partial \bar{x}} & -\frac{\partial y^i}{\partial \bar{y}} \\ \frac{\partial x^f}{\partial \bar{x}} & -\frac{\partial x^f}{\partial \bar{y}} \end{pmatrix}
\end{aligned} \tag{41}
$$

where:

$$
A = \frac{-1}{\left(\frac{\partial x^i}{\partial p_i} - \frac{\partial y^i}{\partial p_i}\right)\left(\frac{\partial x^f}{\partial p_f} - \frac{\partial y^f}{\partial p_f}\right) - \left(\frac{\partial x^i}{\partial p_f} - \frac{\partial y^i}{\partial p_f}\right)\left(\frac{\partial x^f}{\partial p_i} - \frac{\partial y^f}{\partial p_i}\right)}
$$

Using the expressions derived above:

$$
\begin{aligned}
\left(\frac{\partial x^i}{\partial p_i} - \frac{\partial y^i}{\partial p_i}\right) \quad \cdot \quad & \left(\frac{\partial x^f}{\partial p_f} - \frac{\partial y^f}{\partial p_f}\right) = \\
& \left(\int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta, p_i, \bar{x})}{\partial p_i} \cdot g_d(\theta)d\theta + \bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i} - \int_0^{\underline{\theta}^u} \frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i} \cdot g_u(\theta)d\theta - \bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right) \cdot \\
& \left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f} + \int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f} \cdot g_d(\theta)d\theta + y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f} - \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f} \cdot g_u(\theta)d\theta\right) \\
= \quad & \left(\bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i}\right)\left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f}\right) + \left(\bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right)\left(x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f}\right) + \\
& \left(\bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i}\right)\left(y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f}\right) - \left(\bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right)\left(y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f}\right) + integrals \\
= \quad & \left(\bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i}\right)\left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f}\right) - \left(\bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right)\left(y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f}\right) + \\
& \left(g_u(\bar{\theta}^u) g_d(\bar{\theta}^d)\right)\left[\left(\frac{\partial \bar{\theta}^u}{\partial p_i}\frac{\partial \bar{\theta}^d}{\partial p_f}\right)\left(\bar{y} \cdot x^f(\bar{\theta}^d, pf, \bar{x})\right) + \left(\frac{\partial \bar{\theta}^d}{\partial p_i}\frac{\partial \bar{\theta}^u}{\partial p_f}\right)\left(\bar{x} \cdot y^f(\bar{\theta}^u, p_f, \bar{y})\right)\right] + \\
& \left(\int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta, p_i, \bar{x})}{\partial p_i} \cdot g_d(\theta)d\theta - \int_0^{\underline{\theta}^u} \frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i} \cdot g_u(\theta)d\theta\right) \cdot \\
& \left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f} + \int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f} \cdot g_d(\theta)d\theta + y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f} - \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f} \cdot g_u(\theta)d\theta\right) + \\
& \left(\bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i} - \bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right) \cdot \\
& \left(\int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f} \cdot g_d(\theta)d\theta - \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f} \cdot g_u(\theta)d\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\partial x^i}{\partial p_f} - \frac{\partial y^i}{\partial p_f}\right) \quad \cdot \quad & \left(\frac{\partial x^f}{\partial p_i} - \frac{\partial y^f}{\partial p_i}\right) = \\
& \left(\bar{x} g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial p_f} - \bar{y} g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial p_f}\right) \cdot \left(y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial p_i} - x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial p_i}\right) \\
= \quad & \left(\bar{x} g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial p_f} y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial p_i}\right) - \left(\bar{x} g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial p_f} x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial p_i}\right) - \left(\bar{y} g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial p_f} y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial p_i}\right) + \left(\bar{y} g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial p_f} x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial p_i}\right) \\
= \quad & \left(g_d(\bar{\theta}^d) g_u(\bar{\theta}^u)\right) \left[\left(\frac{\partial \bar{\theta}^d}{\partial p_f} \frac{\partial \bar{\theta}^u}{\partial p_i}\right) \left(\bar{x} \cdot y^f(\bar{\theta}^u, p_f, \bar{y})\right) + \left(\frac{\partial \bar{\theta}^u}{\partial p_f} \frac{\partial \bar{\theta}^d}{\partial p_i}\right) \left(\bar{y} \cdot x^f(\bar{\theta}^d, pf, \bar{x})\right)\right] - \left(\bar{x} g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial p_f} x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial p_i}\right) - \left(\bar{y} g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial p_f} y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial p_i}\right)
\end{aligned}
$$

Then:

$$
\begin{aligned}
\left(\frac{\partial x^i}{\partial p_i}-\frac{\partial y^i}{\partial p_i}\right)\left(\frac{\partial x^f}{\partial p_f}-\frac{\partial y^f}{\partial p_f}\right) \quad - \quad & \left(\frac{\partial x^i}{\partial p_f}-\frac{\partial y^i}{\partial p_f}\right)\left(\frac{\partial x^f}{\partial p_i}-\frac{\partial y^f}{\partial p_i}\right)= \\
& \left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_i}\right)\left(-x^f(\bar{\theta}^d,pf,\bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_f}\right)-\left(\bar{y}g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial p_i}\right)\left(y^f(\bar{\theta}^u,p_f,\bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial p_f}\right)+ \\
& \left(g_u(\bar{\theta}^u)g_d(\bar{\theta}^d)\right)\left[\left(\frac{\partial\bar{\theta}^u}{\partial p_i}\frac{\partial\bar{\theta}^d}{\partial p_f}\right)\left(\bar{y}\cdot x^f(\bar{\theta}^d,pf,\bar{x})\right)+\left(\frac{\partial\bar{\theta}^d}{\partial p_i}\frac{\partial\bar{\theta}^u}{\partial p_f}\right)\left(\bar{x}\cdot y^f(\bar{\theta}^u,p_f,\bar{y})\right)\right]+integrals \\
& -\left(g_d(\bar{\theta}^d)g_u(\bar{\theta}^u)\right)\left[\left(\frac{\partial\bar{\theta}^d}{\partial p_f}\frac{\partial\bar{\theta}^u}{\partial p_i}\right)\left(\bar{x}\cdot y^f(\bar{\theta}^u,p_f,\bar{y})\right)+\left(\frac{\partial\bar{\theta}^u}{\partial p_f}\frac{\partial\bar{\theta}^d}{\partial p_i}\right)\left(\bar{y}\cdot x^f(\bar{\theta}^d,pf,\bar{x})\right)\right] \\
& +\left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_f}x^f(\bar{\theta}^d,pf,\bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_i}\right)+\left(\bar{y}g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial p_f}y^f(\bar{\theta}^u,p_f,\bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial p_i}\right) \\
= \quad & \left(g_u(\bar{\theta}^u)g_d(\bar{\theta}^d)\right)\left[\left(\frac{\partial\bar{\theta}^u}{\partial p_i}\frac{\partial\bar{\theta}^d}{\partial p_f}\right)\left(\bar{y}\cdot x^f(\bar{\theta}^d,pf,\bar{x})\right)+\left(\frac{\partial\bar{\theta}^d}{\partial p_i}\frac{\partial\bar{\theta}^u}{\partial p_f}\right)\left(\bar{x}\cdot y^f(\bar{\theta}^u,p_f,\bar{y})\right)\right]+integrals \\
& -\left(g_d(\bar{\theta}^d)g_u(\bar{\theta}^u)\right)\left[\left(\frac{\partial\bar{\theta}^d}{\partial p_f}\frac{\partial\bar{\theta}^u}{\partial p_i}\right)\left(\bar{x}\cdot y^f(\bar{\theta}^u,p_f,\bar{y})\right)+\left(\frac{\partial\bar{\theta}^u}{\partial p_f}\frac{\partial\bar{\theta}^d}{\partial p_i}\right)\left(\bar{y}\cdot x^f(\bar{\theta}^d,pf,\bar{x})\right)\right] \\
= \quad & \underbrace{\left(g_u(\bar{\theta}^u)g_d(\bar{\theta}^d)\right)}_{\geq 0}\underbrace{\left[\left(\frac{\partial\bar{\theta}^u}{\partial p_i}\frac{\partial\bar{\theta}^d}{\partial p_f}\right)-\left(\frac{\partial\bar{\theta}^d}{\partial p_i}\frac{\partial\bar{\theta}^u}{\partial p_f}\right)\right]}_{(*)}\underbrace{\left(\bar{y}\cdot x^f(\bar{\theta}^d,pf,\bar{x})-\bar{x}\cdot y^f(\bar{\theta}^u,p_f,\bar{y})\right)}_{(**)}+ \\
& \left(\int_0^{\underline{\theta}^d}\frac{\partial x^i(\theta,p_i,\bar{x})}{\partial p_i}\cdot g_d(\theta)d\theta-\int_0^{\underline{\theta}^u}\frac{\partial y^i(\theta,p_i,\bar{y})}{\partial p_i}\cdot g_u(\theta)d\theta\right)\cdot \\
& \left(-x^f(\bar{\theta}^d,pf,\bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_f}+\int_{\bar{\theta}^d}^{\theta^d_{max}}\frac{\partial x^f(\theta,pf,\bar{x})}{\partial p_f}\cdot g_d(\theta)d\theta+y^f(\bar{\theta}^u,p_f,\bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial p_f}-\int_{\bar{\theta}^u}^{\theta^u_{max}}\frac{\partial y^f(\theta,p_f,\bar{y})}{\partial p_f}\cdot g_u(\theta)d\theta\right)+ \\
& \left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_i}-\bar{y}g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial p_i}\right)\cdot \\
& \left(\int_{\bar{\theta}^d}^{\theta^d_{max}}\frac{\partial x^f(\theta,pf,\bar{x})}{\partial p_f}\cdot g_d(\theta)d\theta-\int_{\bar{\theta}^u}^{\theta^u_{max}}\frac{\partial y^f(\theta,p_f,\bar{y})}{\partial p_f}\cdot g_u(\theta)d\theta\right)
\end{aligned}
$$

The terms involving integrals are all positive, so this expression is positive as long as (∗) and (∗∗) are positive. DePaula and Scheinkman show that (∗) is positive. However, they have the term (∗∗) wrong, as they assume that the supply of the informal good is always θ. Therefore, this term is positive if, and only if:

$$\frac{x^f(\bar{\theta}^d, p_f, \bar{x})}{\bar{x}} \geq \frac{y^f(\bar{\theta}^u, p_f, \bar{y})}{\bar{y}} \tag{42}$$

This condition compares the level of constrainedness in either sector. When the downstream sector is more constrained by $\bar{x}$ or, equivalently, when the difference between the marginal formal firm and the marginal informal firm is larger, this condition holds. En realidad solo se puede decir algo cuando esta condicion se cumple. Si no, el denominador es ambiguo y por lo tanto las derivadas.

$$
\begin{aligned}
\left(\frac{\partial y^f}{\partial p_f}-\frac{\partial x^f}{\partial p_f}\right)\left(\frac{\partial x^i}{\partial \bar{x}}\right) \quad + \quad & \left(\frac{\partial x^i}{\partial p_f}-\frac{\partial y^i}{\partial p_f}\right)\left(\frac{\partial x^f}{\partial \bar{x}}\right) = \\
& \left(x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f} - \int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f}\cdot g_d(\theta)d\theta - y^f(\bar{\theta}^u, p_f, \bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f} + \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f}\cdot g_u(\theta)d\theta\right)\cdot \\
& \left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}} + \int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta\right) + \\
& \left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f} - \bar{y}g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f}\right)\cdot\left(-x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right) \\
= \quad & \left(x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f}\right)\left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right) - \left(y^f(\bar{\theta}^u, p_f, \bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f}\right)\left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right) + \\
& -\left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f}\right)\cdot\left(x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right) + \left(\bar{y}g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f}\right)\cdot\left(x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right) + integrals \\
= \quad & \underbrace{\left(g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right)}_{<0}\underbrace{\left(\bar{y}\cdot x^f(\bar{\theta}^d, pf, \bar{x}) - \bar{x}\cdot y^f(\bar{\theta}^u, p_f, \bar{y})\right)}_{>0 \text{ if } (42) \text{ holds}} + \\
& \left(-\int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f}\cdot g_d(\theta)d\theta + \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f}\cdot g_u(\theta)d\theta\right)\cdot\left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}} + \int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta\right) + \\
& \left(\int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta\right)\cdot\left(x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f} - y^f(\bar{\theta}^u, p_f, \bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f}\right)
\end{aligned}
$$

Thus, the derivative is:

$$\frac{\partial p_i}{\partial \bar{x}} = \frac{\left(g_u(\bar{\theta}^u) g_d(\bar{\theta}^d)\right)\left(\frac{\partial \bar{\theta}^u}{\partial p_f}\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right)\left(\bar{y}\cdot x^f(\bar{\theta}^d, pf, \bar{x}) - \bar{x}\cdot y^f(\bar{\theta}^u, p_f, \bar{y})\right) +}{\left(g_u(\bar{\theta}^u) g_d(\bar{\theta}^d)\right)\left[\left(\frac{\partial \bar{\theta}^u}{\partial p_i}\frac{\partial \bar{\theta}^d}{\partial p_f}\right) - \left(\frac{\partial \bar{\theta}^d}{\partial p_i}\frac{\partial \bar{\theta}^u}{\partial p_f}\right)\right]\left(\bar{y}\cdot x^f(\bar{\theta}^d, pf, \bar{x}) - \bar{x}\cdot y^f(\bar{\theta}^u, p_f, \bar{y})\right)} \cdots$$

$$\cdots \frac{+\left(-\int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f}\cdot g_d(\theta)d\theta + \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f}\cdot g_u(\theta)d\theta\right)\cdot\left(\bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}} + \int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta\right) + \left(\int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta\right)\cdot\left(x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f} - y^f(\bar{\theta}^u, p_f, \bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f}\right)}{+\left(\int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta, p_i, \bar{x})}{\partial p_i}\cdot g_d(\theta)d\theta - \int_0^{\underline{\theta}^u} \frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i}\cdot g_u(\theta)d\theta\right)\cdot\left(-x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f} + \int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f}\cdot g_d(\theta)d\theta + y^f(\bar{\theta}^u, p_f, \bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f} - \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f}\cdot g_u(\theta)d\theta\right) + \left(\bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i} - \bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right)\cdot\left(\int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f}\cdot g_d(\theta)d\theta - \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f}\cdot g_u(\theta)d\theta\right)}$$

Mirar efecto directo/indirecto:

$$\frac{\partial \bar{\theta}^d}{\partial \bar{x}} = \left(\frac{\partial \bar{\theta}^d}{\partial p_i}\right)\underbrace{\left(g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right)}_{<0}\underbrace{\left(\bar{y}\cdot x^f(\bar{\theta}^d, pf, \bar{x}) - \bar{x}\cdot y^f(\bar{\theta}^u, p_f, \bar{y})\right)}_{>0 \text{ if (42) holds}} + > 0$$

$$\left(\frac{\partial \bar{\theta}^d}{\partial p_i}\right)\left(-\int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f}\cdot g_d(\theta)d\theta + \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f}\cdot g_u(\theta)d\theta\right)\cdot\left(\bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}} + \int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta\right) + < 0$$

$$\left(\frac{\partial \bar{\theta}^d}{\partial p_i}\right)\left(\int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta\right)\cdot\left(x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_f} - y^f(\bar{\theta}^u, p_f, \bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_f}\right) < 0$$

$$+\left(\frac{\partial \bar{\theta}^d}{\partial p_f}\right)\left(g_u(\bar{\theta}^u) g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\frac{\partial \bar{\theta}^u}{\partial p_i}\right)\underbrace{\left(\bar{x}\cdot y^f(\bar{\theta}^u, p_f, \bar{y}) - \bar{y}\cdot x^f(\bar{\theta}^d, pf, \bar{x})\right)}_{<0 \text{ if condition (42) holds}} +$$

$$+\left(\frac{\partial \bar{\theta}^d}{\partial p_f}\right)\left(\int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta\right)\cdot\left(-x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i} + y^f(\bar{\theta}^u, p_f, \bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right)\cdot$$

$$+\left(\frac{\partial \bar{\theta}^d}{\partial p_f}\right)\left(\int_0^{\underline{\theta}^u} \frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i}\cdot g_u(\theta)d\theta - \int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta, p_i, \bar{x})}{\partial p_i}\cdot g_d(\theta)d\theta\right)\cdot\left(-x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right)$$

Also:

$$-\left(\frac{\partial y^f}{\partial p_f}-\frac{\partial x^f}{\partial p_f}\right)\left(\frac{\partial y^i}{\partial \bar{y}}\right) \quad - \quad \left(\frac{\partial x^i}{\partial p_f}-\frac{\partial y^i}{\partial p_f}\right)\left(\frac{\partial y^f}{\partial \bar{y}}\right)=$$

$$-\left(x^f(\bar{\theta}^d,pf,\bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_f}-\int_{\bar{\theta}^d}^{\theta^d_{max}}\frac{\partial x^f(\theta,pf,\bar{x})}{\partial p_f}\cdot g_d(\theta)d\theta-y^f(\bar{\theta}^u,p_f,\bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial p_f}+\int_{\bar{\theta}^u}^{\theta^u_{max}}\frac{\partial y^f(\theta,p_f,\bar{y})}{\partial p_f}\cdot g_u(\theta)d\theta\right)\cdot$$

$$\left(\bar{y}g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial\bar{y}}+\int_{\underline{\theta}^u}^{\bar{\theta}^u}g_u(\theta)d\theta\right)-\left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_f}-\bar{y}g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial p_f}\right)\cdot\left(-y^f(\bar{\theta}^u,p_f,\bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial\bar{y}}\right)$$

$$= \quad -\left(x^f(\bar{\theta}^d,pf,\bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_f}\right)\left(\bar{y}g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial\bar{y}}\right)+\left(y^f(\bar{\theta}^u,p_f,\bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial p_f}\right)\left(\bar{y}g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial\bar{y}}\right)$$

$$+\left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_f}\right)\left(y^f(\bar{\theta}^u,p_f,\bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial\bar{y}}\right)-\left(\bar{y}g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial p_f}\right)\left(y^f(\bar{\theta}^u,p_f,\bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial\bar{y}}\right)+integrals$$

$$= \quad -\left(x^f(\bar{\theta}^d,pf,\bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_f}\right)\left(\bar{y}g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial\bar{y}}\right)+\left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_f}\right)\left(y^f(\bar{\theta}^u,p_f,\bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial\bar{y}}\right)+integrals$$

$$= \quad \underbrace{\left(g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_f}g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial\bar{y}}\right)}_{>0}\underbrace{\left(\bar{x}\cdot y^f(\bar{\theta}^u,p_f,\bar{y})-\bar{y}\cdot x^f(\bar{\theta}^d,pf,\bar{x})\right)}_{<0 \text{ if condition (42) holds}}+$$

$$\left(\int_{\bar{\theta}^d}^{\theta^d_{max}}\frac{\partial x^f(\theta,pf,\bar{x})}{\partial p_f}\cdot g_d(\theta)d\theta-\int_{\bar{\theta}^u}^{\theta^u_{max}}\frac{\partial y^f(\theta,p_f,\bar{y})}{\partial p_f}\cdot g_u(\theta)d\theta\right)\cdot\left(\bar{y}g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial\bar{y}}+\int_{\underline{\theta}^u}^{\bar{\theta}^u}g_u(\theta)d\theta\right)-$$

$$\left(\int_{\underline{\theta}^u}^{\bar{\theta}^u}g_u(\theta)d\theta\right)\cdot\left(x^f(\bar{\theta}^d,pf,\bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_f}-y^f(\bar{\theta}^u,p_f,\bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial p_f}\right)$$

So the derivative $\frac{\partial p_i}{\partial \bar{y}}$ is negative if condition (42) holds:

$$\frac{\partial p_i}{\partial\bar{y}}=\frac{\left(g_d(\bar{\theta}^d)\frac{\partial\bar{\theta}^d}{\partial p_f}g_u(\bar{\theta}^u)\frac{\partial\bar{\theta}^u}{\partial\bar{y}}\right)\left(\bar{x}\cdot y^f(\bar{\theta}^u,p_f,\bar{y})-\bar{y}\cdot x^f(\bar{\theta}^d,pf,\bar{x})\right)+integrals}{\left(g_u(\bar{\theta}^u)g_d(\bar{\theta}^d)\right)\left[\left(\frac{\partial\bar{\theta}^u}{\partial p_i}\frac{\partial\bar{\theta}^d}{\partial p_f}\right)-\left(\frac{\partial\bar{\theta}^d}{\partial p_i}\frac{\partial\bar{\theta}^u}{\partial p_f}\right)\right]\left(\bar{y}\cdot x^f(\bar{\theta}^d,pf,\bar{x})-\bar{x}\cdot y^f(\bar{\theta}^u,p_f,\bar{y})\right)+integrals}<0$$

Note that condition (42) is a sufficient condition, but not a necessary condition (given the integral terms in the above expressions).

The numerator of $\frac{\partial p_i}{\partial \bar{x}} - \frac{\partial p_i}{\partial \bar{y}}$ is given by:

$$
\begin{aligned}
\frac{\partial p_i}{\partial \bar{x}} - \frac{\partial p_i}{\partial \bar{y}} \quad = \quad & \left(g_u(\bar{\theta}^u) g_d(\bar{\theta}^d)\right) \left(\frac{\partial \bar{\theta}^u}{\partial p_f} \frac{\partial \bar{\theta}^d}{\partial \bar{x}} + \frac{\partial \bar{\theta}^d}{\partial p_f} \frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right) \left(\bar{y} \cdot x^f(\bar{\theta}^d, pf, \bar{x}) - \bar{x} \cdot y^f(\bar{\theta}^u, p_f, \bar{y})\right) + \\
& \left(-\int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f} \cdot g_d(\theta) d\theta + \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f} \cdot g_u(\theta) d\theta\right) \cdot \left(\bar{x} g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial \bar{x}} + \int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta) d\theta\right) + \\
& \left(\int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta) d\theta\right) \cdot \left(x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial p_f} - y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial p_f}\right) \\
& - \left(\int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f} \cdot g_d(\theta) d\theta - \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f} \cdot g_u(\theta) d\theta\right) \cdot \left(\bar{y} g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial \bar{y}} + \int_{\underline{\theta}^u}^{\bar{\theta}^u} g_u(\theta) d\theta\right) \\
& + \left(\int_{\underline{\theta}^u}^{\bar{\theta}^u} g_u(\theta) d\theta\right) \cdot \left(x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial p_f} - y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial p_f}\right) \\
= \quad & \left(g_u(\bar{\theta}^u) g_d(\bar{\theta}^d)\right) \left(\frac{\partial \bar{\theta}^u}{\partial p_f} \frac{\partial \bar{\theta}^d}{\partial \bar{x}} + \frac{\partial \bar{\theta}^d}{\partial p_f} \frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right) \left(\bar{y} \cdot x^f(\bar{\theta}^d, pf, \bar{x}) - \bar{x} \cdot y^f(\bar{\theta}^u, p_f, \bar{y})\right) + \\
& \left(-\int_{\bar{\theta}^d}^{\theta^d_{max}} \frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f} \cdot g_d(\theta) d\theta + \int_{\bar{\theta}^u}^{\theta^u_{max}} \frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f} \cdot g_u(\theta) d\theta\right) \cdot \left(\bar{x} g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial \bar{x}} + \int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta) d\theta + \bar{y} g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial \bar{y}} + \int_{\underline{\theta}^u}^{\bar{\theta}^u} g_u(\theta) d\theta\right) + \\
& \left(\int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta) d\theta + \int_{\underline{\theta}^u}^{\bar{\theta}^u} g_u(\theta) d\theta\right) \cdot \left(x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d) \frac{\partial \bar{\theta}^d}{\partial p_f} - y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u) \frac{\partial \bar{\theta}^u}{\partial p_f}\right)
\end{aligned}
$$

The numerator of $\frac{\partial p_f}{\partial \bar{x}} - \frac{\partial p_f}{\partial \bar{y}}$ is given by:

$$
\begin{aligned}
\frac{\partial p_f}{\partial \bar{x}} - \frac{\partial p_f}{\partial \bar{y}} \quad = \quad & \left(g_u(\bar{\theta}^u) g_d(\bar{\theta}^d)\right)\left(\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\frac{\partial \bar{\theta}^u}{\partial p_i} + \frac{\partial \bar{\theta}^d}{\partial p_i}\frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right)\left(\bar{x}\cdot y^f(\bar{\theta}^u, p_f, \bar{y}) - \bar{y}\cdot x^f(\bar{\theta}^d, pf, \bar{x})\right) \\
& + \left(\int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta + \int_{\underline{\theta}^u}^{\bar{\theta}^u} g_u(\theta)d\theta\right)\cdot\left(-x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i} + y^f(\bar{\theta}^u, p_f, \bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right)\cdot \\
& + \left(\int_0^{\underline{\theta}^u}\frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i}\cdot g_u(\theta)d\theta - \int_0^{\underline{\theta}^d}\frac{\partial x^i(\theta, p_i, \bar{x})}{\partial p_i}\cdot g_d(\theta)d\theta\right)\cdot\left(-x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}} - y^f(\bar{\theta}^u, p_f, \bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \bar{\theta}^d}{\partial \bar{x}} - \frac{\partial \bar{\theta}^d}{\partial \bar{y}} \quad = \quad & \underbrace{\left(g_u(\bar{\theta}^u) g_d(\bar{\theta}^d)\right)\left(\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right)\left(\frac{\partial \bar{\theta}^d}{\partial p_i}\frac{\partial \bar{\theta}^u}{\partial p_f} - \frac{\partial \bar{\theta}^u}{\partial p_i}\frac{\partial \bar{\theta}^d}{\partial p_f}\right)\left(\bar{y}\cdot x^f(\bar{\theta}^d, pf, \bar{x}) - \bar{x}\cdot y^f(\bar{\theta}^u, p_f, \bar{y})\right)}_{<0} + \underbrace{\frac{\partial \bar{\theta}^d}{\partial \bar{x}}}_{>0} \\
& + \underbrace{\left(\frac{\partial \bar{\theta}^d}{\partial p_i}\right)\left(-\int_{\bar{\theta}^d}^{\theta^d_{max}}\frac{\partial x^f(\theta, pf, \bar{x})}{\partial p_f}\cdot g_d(\theta)d\theta + \int_{\bar{\theta}^u}^{\theta^u_{max}}\frac{\partial y^f(\theta, p_f, \bar{y})}{\partial p_f}\cdot g_u(\theta)d\theta\right)\cdot\left(\bar{x}g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}} + \int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta + \bar{y}g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}} + \int_{\underline{\theta}^u}^{\bar{\theta}^u} g_u(\theta)d\theta\right)}_{<0} \\
& + \underbrace{\left(\frac{\partial \bar{\theta}^d}{\partial p_f}\right)\left(\int_0^{\underline{\theta}^u}\frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i}\cdot g_u(\theta)d\theta - \int_0^{\underline{\theta}^d}\frac{\partial x^i(\theta, p_i, \bar{x})}{\partial p_i}\cdot g_d(\theta)d\theta\right)\cdot\left(-x^f(\bar{\theta}^d, pf, \bar{x})\cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}} - y^f(\bar{\theta}^u, p_f, \bar{y})\cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right)}_{<0} \\
& + \underbrace{\left(\int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta)d\theta + \int_{\underline{\theta}^u}^{\bar{\theta}^u} g_u(\theta)d\theta\right)\cdot\left(y^f(\bar{\theta}^u, p_f, \bar{y})\cdot g_u(\bar{\theta}^u)\right)\left(\frac{\partial \bar{\theta}^u}{\partial p_i}\frac{\partial \bar{\theta}^d}{\partial p_f} - \frac{\partial \bar{\theta}^u}{\partial p_f}\frac{\partial \bar{\theta}^d}{\partial p_i}\right)}_{>0} \qquad (43)
\end{aligned}
$$

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$$\left(\int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta,p_i,\bar{x})}{\partial p_i}\cdot g_d(\theta)d\theta - \int_0^{\underline{\theta}^u}\frac{\partial y^i(\theta,p_i,\bar{y})}{\partial p_i}\cdot g_u(\theta)d\theta\right)\cdot$$

$$\left(-x^f(\bar\theta^d,pf,\bar x)\cdot g_d(\bar\theta^d)\frac{\partial\bar\theta^d}{\partial p_f} + \int_{\bar\theta^d}^{\theta^d_{max}} \frac{\partial x^f(\theta,pf,\bar x)}{\partial p_f}\cdot g_d(\theta)d\theta + y^f(\bar\theta^u,p_f,\bar y)\cdot g_u(\bar\theta^u)\frac{\partial\bar\theta^u}{\partial p_f} - \int_{\bar\theta^u}^{\theta^u_{max}}\frac{\partial y^f(\theta,p_f,\bar y)}{\partial p_f}\cdot g_u(\theta)d\theta\right)+$$

$$\left(\bar x g_d(\bar\theta^d)\frac{\partial\bar\theta^d}{\partial p_i} - \bar y g_u(\bar\theta^u)\frac{\partial \bar\theta^u}{\partial p_i}\right)\cdot\left(\int_{\bar\theta^d}^{\theta^d_{max}}\frac{\partial x^f(\theta,pf,\bar x)}{\partial p_f}\cdot g_d(\theta)d\theta - \int_{\bar\theta^u}^{\theta^u_{max}}\frac{\partial y^f(\theta,p_f,\bar y)}{\partial p_f}\cdot g_u(\theta)d\theta\right)>0$$

$$\left(\frac{\partial\bar\theta^d}{\partial p_i}\right)\left(\int_{\bar\theta^u}^{\theta^u_{max}}\frac{\partial y^f(\theta,p_f,\bar y)}{\partial p_f}\cdot g_u(\theta)d\theta - \int_{\bar\theta^d}^{\theta^d_{max}}\frac{\partial x^f(\theta,pf,\bar x)}{\partial p_f}\cdot g_d(\theta)d\theta\right)\cdot\left(\bar x g_d(\bar\theta^d)\frac{\partial\bar\theta^d}{\partial\bar x} + \int_{\underline{\theta}^d}^{\bar\theta^d} g_d(\theta)d\theta + \bar y g_u(\bar\theta^u)\frac{\partial\bar\theta^u}{\partial\bar y} + \int_{\underline{\theta}^u}^{\bar\theta^u} g_u(\theta)d\theta\right) < 0$$

$$+\left(\frac{\partial\bar\theta^d}{\partial p_f}\right)\left(\int_0^{\underline{\theta}^u}\frac{\partial y^i(\theta,p_i,\bar y)}{\partial p_i}\cdot g_u(\theta)d\theta - \int_0^{\underline{\theta}^d}\frac{\partial x^i(\theta,p_i,\bar x)}{\partial p_i}\cdot g_d(\theta)d\theta\right)\cdot\left(-x^f(\bar\theta^d,pf,\bar x)\cdot g_d(\bar\theta^d)\frac{\partial\bar\theta^d}{\partial\bar x} - y^f(\bar\theta^u,p_f,\bar y)\cdot g_u(\bar\theta^u)\frac{\partial\bar\theta^u}{\partial\bar y}\right) < 0$$

$$+\left(\frac{\partial\bar\theta^d}{\partial p_i}\right)\left(\int_{\underline{\theta}^d}^{\bar\theta^d} g_d(\theta)d\theta + \int_{\underline{\theta}^u}^{\bar\theta^u} g_u(\theta)d\theta\right)\cdot\left(-y^f(\bar\theta^u,p_f,\bar y)\cdot g_u(\bar\theta^u)\frac{\partial\bar\theta^u}{\partial p_f}\right) < 0$$

$$+\left(\frac{\partial\bar\theta^d}{\partial p_f}\right)\left(\int_{\underline{\theta}^d}^{\bar\theta^d} g_d(\theta)d\theta + \int_{\underline{\theta}^u}^{\bar\theta^u} g_u(\theta)d\theta\right)\cdot\left(y^f(\bar\theta^u,p_f,\bar y)\cdot g_u(\bar\theta^u)\frac{\partial\bar\theta^u}{\partial p_i}\right) > 0$$

$$\frac{\partial p_f}{\partial \bar{x}} = (-A)\left[\cdot\left(\frac{\partial x^f}{\partial p_i} - \frac{\partial y^f}{\partial p_i}\right)\left(\frac{\partial x^i}{\partial \bar{x}}\right) + \left(\frac{\partial y^i}{\partial p_i} - \frac{\partial x^i}{\partial p_i}\right)\left(\frac{\partial x^f}{\partial \bar{x}}\right)\right]$$

So, the derivative is:

$$\begin{aligned}
& \left(\frac{\partial x^f}{\partial p_i} - \frac{\partial y^f}{\partial p_i}\right)\left(\frac{\partial x^i}{\partial \bar{x}}\right) + \left(\frac{\partial y^i}{\partial p_i} - \frac{\partial x^i}{\partial p_i}\right)\left(\frac{\partial x^f}{\partial \bar{x}}\right) = \\
& \left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i} + y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right) \cdot \\
& \left(\bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}} + \int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta) d\theta\right) + \\
& \left(\int_0^{\underline{\theta}^u} \frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i} \cdot g_u(\theta) d\theta + \bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i} - \int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta, p_i, \bar{x})}{\partial p_i} \cdot g_d(\theta) d\theta - \bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i}\right) \cdot \left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right) \\
= \; & \left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i}\right)\left(\bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right) + \left(y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right)\left(\bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right) + \\
& \left(\bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right)\left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right) + \left(-\bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i}\right)\left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right) + integrals \\
= \; & \underbrace{\left(g_u(\bar{\theta}^u) g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\frac{\partial \bar{\theta}^u}{\partial p_i}\right)}_{>0}\underbrace{\left(\bar{x} \cdot y^f(\bar{\theta}^u, p_f, \bar{y}) - \bar{y} \cdot x^f(\bar{\theta}^d, pf, \bar{x})\right)}_{<0 \text{ if condition (42) holds}} + \\
& \left(\int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta) d\theta\right) \cdot \left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i} + y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right) \cdot \\
& + \left(\int_0^{\underline{\theta}^u} \frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i} \cdot g_u(\theta) d\theta - \int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta, p_i, \bar{x})}{\partial p_i} \cdot g_d(\theta) d\theta\right) \cdot \left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\right)
\end{aligned}$$

$$
\begin{aligned}
& \left(\frac{\partial y^f}{\partial p_i} - \frac{\partial x^f}{\partial p_i}\right)\left(\frac{\partial y^i}{\partial \bar{y}}\right) + \left(\frac{\partial x^i}{\partial p_i} - \frac{\partial y^i}{\partial p_i}\right)\left(\frac{\partial y^f}{\partial \bar{y}}\right) \\
= \quad & \left(-y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i} + x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i}\right)\left(\bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}} + \int_{\underline{\theta}^u}^{\bar{\theta}^u} g_u(\theta) d\theta\right) + \\
& \left(\int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta, p_i, \bar{x})}{\partial p_i} \cdot g_d(\theta) d\theta + \bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i} - \int_0^{\underline{\theta}^u} \frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i} \cdot g_u(\theta) d\theta - \bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right)\left(-y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right) \\
= \quad & \left(-y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right)\left(\bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right) + \left(x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i}\right)\left(\bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right) + \\
& \left(\bar{x} g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i}\right)\left(-y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right) + \left(-\bar{y} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right)\left(-y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right) + integrals \\
= \quad & \underbrace{\left(g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i} g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right)}_{<0}\underbrace{\left(\bar{y} \cdot x^f(\bar{\theta}^d, pf, \bar{x}) - \bar{x} \cdot y^f(\bar{\theta}^u, p_f, \bar{y})\right)}_{>0 \text{ if condition (42) holds}} + \\
& \left(\int_{\underline{\theta}^u}^{\bar{\theta}^u} g_u(\theta) d\theta\right) \cdot \left(-y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i} + x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i}\right) + \left(\int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta, p_i, \bar{x})}{\partial p_i} \cdot g_d(\theta) d\theta - \int_0^{\underline{\theta}^u} \frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i} \cdot g_u(\theta) d\theta\right)()
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial p_f}{\partial \bar{x}} - \frac{\partial p_f}{\partial \bar{y}} \quad = \quad & \left(g_u(\bar{\theta}^u) g_d(\bar{\theta}^d)\right)\left(\frac{\partial \bar{\theta}^d}{\partial \bar{x}}\frac{\partial \bar{\theta}^u}{\partial p_i} + \frac{\partial \bar{\theta}^d}{\partial p_i}\frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right)\left(\bar{x} \cdot y^f(\bar{\theta}^u, p_f, \bar{y}) - \bar{y} \cdot x^f(\bar{\theta}^d, pf, \bar{x})\right) \\
& \left(\int_{\underline{\theta}^d}^{\bar{\theta}^d} g_d(\theta) d\theta + \int_{\underline{\theta}^u}^{\bar{\theta}^u} g_u(\theta) d\theta\right) \cdot \left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial p_i} + y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial p_i}\right) \cdot \\
& + \left(\int_0^{\underline{\theta}^u} \frac{\partial y^i(\theta, p_i, \bar{y})}{\partial p_i} \cdot g_u(\theta) d\theta - \int_0^{\underline{\theta}^d} \frac{\partial x^i(\theta, p_i, \bar{x})}{\partial p_i} \cdot g_d(\theta) d\theta\right) \cdot \left(-x^f(\bar{\theta}^d, pf, \bar{x}) \cdot g_d(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{x}} - y^f(\bar{\theta}^u, p_f, \bar{y}) \cdot g_u(\bar{\theta}^u)\frac{\partial \bar{\theta}^u}{\partial \bar{y}}\right)
\end{aligned}
$$

Using the implicit function theorem as in (41):

$$\frac{\partial p_i}{\partial \bar{x}} = \frac{\overbrace{\left(\frac{\partial y^f}{\partial p_f} - \frac{\partial x^f}{\partial p_f}\right)}^{>0}\overbrace{\left(\frac{\partial x^i}{\partial \bar{x}}\right)}^{>0} + \overbrace{\left(\frac{\partial x^i}{\partial p_f} - \frac{\partial y^i}{\partial p_f}\right)}^{>0}\overbrace{\left(\frac{\partial x^f}{\partial \bar{x}}\right)}^{<0}}{\underbrace{\left(\frac{\partial x^i}{\partial p_i} - \frac{\partial y^i}{\partial p_i}\right)}_{<0}\underbrace{\left(\frac{\partial x^f}{\partial p_f} - \frac{\partial y^f}{\partial p_f}\right)}_{<0} - \underbrace{\left(\frac{\partial x^i}{\partial p_f} - \frac{\partial y^i}{\partial p_f}\right)}_{>0}\underbrace{\left(\frac{\partial x^f}{\partial p_i} - \frac{\partial y^f}{\partial p_i}\right)}_{>0}}$$

And:

$$\frac{\partial p_i}{\partial \bar{y}} = \frac{-\overbrace{\left(\frac{\partial y^f}{\partial p_f} - \frac{\partial x^f}{\partial p_f}\right)}^{>0}\overbrace{\left(\frac{\partial y^i}{\partial \bar{y}}\right)}^{>0} - \overbrace{\left(\frac{\partial x^i}{\partial p_f} - \frac{\partial y^i}{\partial p_f}\right)}^{>0}\overbrace{\left(\frac{\partial y^f}{\partial \bar{y}}\right)}^{<0}}{\underbrace{\left(\frac{\partial x^i}{\partial p_i} - \frac{\partial y^i}{\partial p_i}\right)}_{<0}\underbrace{\left(\frac{\partial x^f}{\partial p_f} - \frac{\partial y^f}{\partial p_f}\right)}_{<0} - \underbrace{\left(\frac{\partial x^i}{\partial p_f} - \frac{\partial y^i}{\partial p_f}\right)}_{>0}\underbrace{\left(\frac{\partial x^f}{\partial p_i} - \frac{\partial y^f}{\partial p_i}\right)}_{>0}}$$

So:

$$\begin{aligned} & \frac{\partial p_i}{\partial \bar{x}} \geq -\frac{\partial p_i}{\partial \bar{y}} \\ \iff \quad & \left(\frac{\partial y^f}{\partial p_f} - \frac{\partial x^f}{\partial p_f}\right)\left(\frac{\partial x^i}{\partial \bar{x}}\right) + \left(\frac{\partial x^i}{\partial p_f} - \frac{\partial y^i}{\partial p_f}\right)\left(\frac{\partial x^f}{\partial \bar{x}}\right) \geq \left(\frac{\partial y^f}{\partial p_f} - \frac{\partial x^f}{\partial p_f}\right)\left(\frac{\partial y^i}{\partial \bar{y}}\right) + \left(\frac{\partial x^i}{\partial p_f} - \frac{\partial y^i}{\partial p_f}\right)\left(\frac{\partial y^f}{\partial \bar{y}}\right) \\ \iff \quad & \left(\frac{\partial y^f}{\partial p_f} - \frac{\partial x^f}{\partial p_f}\right)\left(\frac{\partial x^i}{\partial \bar{x}} - \frac{\partial y^i}{\partial \bar{y}}\right) \geq \left(\frac{\partial x^i}{\partial p_f} - \frac{\partial y^i}{\partial p_f}\right)\left(\frac{\partial y^f}{\partial \bar{y}} - \frac{\partial x^f}{\partial \bar{x}}\right) \end{aligned}$$

B Model Parameterization

Profits for formal firms in the upstream sector are described by:

$$\Pi_u^f(\theta_u, p_f, w) = \max_{l} \quad (1-\tau)p_f\theta_u l^{\alpha_u} - wl \tag{44}$$

The optimal demand of labor for formal firms in the upstream sector is given by:

$$l_u^{*,f}(w, p_f, \theta_u) = \left(\frac{(1-\tau)\alpha_u p_f \theta_u}{w}\right)^{\frac{1}{1-\alpha_u}} \tag{45}$$

And replacing the optimal demand for labor in the profits for formal upstream firms:

$$\Pi_u^f(\theta_u, p_f, w) = \left(\frac{(1-\tau)p_f\theta_u}{w^{\alpha_u}}\right)^{\frac{1}{1-\alpha_u}} \left(\alpha_u^{\frac{\alpha_u}{1-\alpha_u}} - \alpha_u^{\frac{1}{1-\alpha_u}}\right) \tag{46}$$

Informal firms in the upstream sector will have profits described by Equation (15):

$$\Pi_u^i(\theta_u, p_i) = \max_{l \leq \bar{l}(\bar{y}_u, \theta_u)} p_i\theta_u l^{\alpha_u} - wl \tag{47}$$

where $\bar{l}(\bar{y}_u, \theta_u)$ is the amount of labor required to produce $\bar{y}_u$:

$$\bar{l}_u(\bar{y}_u, \theta_u) = \left(\frac{\bar{y}_u}{\theta_u}\right)^{\frac{1}{\alpha}} \tag{48}$$

An informal firm will be constrained by size up to a level $\bar{x}$ beyond which she will be observed by the tax authorities. Informal upstream firms that are unconstrained will have demand for labor and profits given by:

$$l_u^{i,n,*}(\alpha_u, \theta_u, w) = \left(\frac{\alpha_u p_i \theta_u}{w}\right)^{\frac{1}{1-\alpha_u}} \tag{49}$$

$$\Pi_u^{i,n}(\theta_u, p_i, w) = \left(\frac{p_i\theta_u}{w^{\alpha_u}}\right)^{\frac{1}{1-\alpha_u}} \left(\alpha_u^{\frac{\alpha_u}{1-\alpha_u}} - \alpha_u^{\frac{1}{1-\alpha_u}}\right) \tag{50}$$

Informal firms that are constrained by size will demand labor as expressed in Equation 48. Their profits are given by:

$$\Pi_u^{i,c}(w, p_i, \theta_u, \bar{x}) = p_i \bar{y}_u - w \left(\frac{\bar{y}_u}{\theta_u}\right)^{\frac{1}{\alpha_u}} \tag{51}$$

The expressions for the informal and the formal supply of the upstream sector (y_u^i, y_u^f) are defined as follows

$$Y_u^i(p_i, p_f, w, \tau, \bar{y}_u) = \underbrace{\int_{\theta_u^{min}}^{\underline{\theta}_u(\bar{y}_u, \alpha_u, w)} \left[\theta_u \left(\frac{\alpha_u p_i}{w}\right)^{\alpha_u}\right]^{\frac{1}{1-\alpha_u}} dF_u(\theta_u)}_{\text{Informal unconstrained supply}}$$
$$+ \underbrace{\bar{y}_u \left[F_u\left(\hat{\theta}_u(p_i, p_f, w, \bar{y}_u, \tau)\right) - F\left(\underline{\theta}_u(\bar{y}_u, \alpha_u, w)\right)\right]}_{\text{Informal constrained supply}} \tag{52}$$

$$Y_u^f(p_i, p_f, w, \tau, \bar{x}) = \int_{\hat{\theta}_u(p_i, p_f, w, \bar{x}, \tau)}^{\theta_u^{max}} \theta_u^{\frac{1}{1-\alpha_u}} \left(\frac{(1-\tau)\alpha_u p_f}{w}\right)^{\frac{\alpha_u}{1-\alpha_u}} dF_u(\theta_u) \tag{53}$$

The pdf of a Pareto distribution Pareto$(\gamma_u, \theta_{u,\min})$ is:

$$f(\theta_u; \gamma_u, \theta_u^{\min}) = \begin{cases} \gamma_u \left(\theta_u^{\min}\right)^{\gamma_u} \theta_u^{-(\gamma_u+1)} & \text{if } \theta_u \geq \theta_u^{\min} \\ 0 & \text{otherwise} \end{cases} \tag{54}$$

The CDF is given by:

$$F(\theta_u; \gamma_u, \theta_{u,\min}) = \begin{cases} 1 - \left(\frac{\theta_u^{min}}{\theta_u}\right)^{\gamma_u} & \text{if } \theta_u \geq \theta_{u,\min} \\ 0 & \text{otherwise} \end{cases} \tag{55}$$

Under the assumption of a Pareto distribution, we can express 52 as:

$$Y_u^i(p_i, p_f, w, \tau, \bar{y}_u) = \left(\frac{\alpha_u p_i}{w}\right)^{\frac{\alpha_u}{1-\alpha_u}} \left(\frac{\gamma_u(1-\alpha_u)}{1+\alpha_u\gamma_u - \gamma_u}\right) (\theta_u^{\min})^{\gamma_u} \theta_u^{\frac{1+\alpha_u\gamma_u-\gamma_u}{1-\alpha_u}} \Big|_{\theta_u^{\min}}^{\underline{\theta}_u(\bar{y}_u, \alpha_w, w)}$$
$$+ \bar{y}_u \left[\left(\frac{\theta_u^{\min}}{\underline{\theta}_u(p_i, p_f, w, \bar{y}_u, \tau)}\right)^{\gamma_u} - \left(\frac{\theta_u^{\min}}{\hat{\theta}_u(p_i, p_f, w, \bar{y}_u, \tau)}\right)^{\gamma_u}\right] \tag{56}$$

Which can be expressed as:

$$
\begin{aligned}
Y_u^i(p_i, p_f, w, \tau, \bar{y}_u) &= (\theta_u^{\min})^{\gamma_u} \left[\left(\frac{\alpha_u p_i}{w} \right)^{\frac{\alpha_u}{1-\alpha_u}} \left(\frac{\gamma_u(1-\alpha_u)}{\gamma_u(1-\alpha_u)-1} \right) \left((\theta_u^{\min})^{\frac{1+\alpha_u\gamma_u-\gamma_u}{1-\alpha_u}} - \underline{\theta}_u(\bar{y}_u)^{\frac{1+\alpha_u\gamma_u-\gamma_u}{1-\alpha_u}} \right) \right. \\
&+ \left. \bar{y}_u \left[\left(\frac{1}{\underline{\theta}_u(p_i, p_f, w, \bar{y}_u, \tau)} \right)^{\gamma_u} - \left(\frac{1}{\hat{\theta}_u(p_i, p_f, w, \bar{y}_u, \tau)} \right)^{\gamma_u} \right] \right] \qquad (57)
\end{aligned}
$$

Similarly, the formal supply can be expressed as:

$$
\begin{aligned}
Y_u^f(p_i, p_f, w, \tau, \bar{y}_u) &= \lim_{\theta_u \to \infty} \left(\frac{(1-\tau)\alpha_u p_f}{w} \right)^{\frac{\alpha_u}{1-\alpha_u}} \left(\frac{\gamma_u(1-\alpha_u)}{1+\alpha_u\gamma_u-\gamma_u} \right) (\theta_u^{\min})^{\gamma_u} \theta^{\frac{1+\alpha\gamma_u-\gamma_u}{1-\alpha}} \Bigg|_{\hat{\theta}_u(p_i, p_f, w, \bar{y}_u, \tau)}^{\theta_u} \\
&= \left(\frac{(1-\tau)\alpha_u p_f}{w} \right)^{\frac{\alpha_u}{1-\alpha_u}} \left(\frac{\gamma_u(1-\alpha_u)}{\gamma_u(1-\alpha_u)-1} \right) \frac{(\theta_u^{\min})^{\gamma_u}}{\hat{\theta}_u(p_i, p_f, \bar{y}, w, \tau)^{\gamma_u(1-\alpha)-1}} \qquad (58)
\end{aligned}
$$

so long as $\gamma_u > \frac{1}{1-\alpha_u}$.

B.1 Downstream

The labor and input demands in the downstream sector are, respectively:

$$
l_d^f(p_f, w, \theta_d) = \left[\theta_d (1-\tau)^{1-\beta_d} \left(\frac{\alpha_d}{w} \right)^{1-\beta_d} \left(\frac{\beta_d}{p_f} \right)^{\beta_d} \right]^{\frac{1}{1-\alpha_d-\beta_d}} \qquad (59)
$$

$$
x_d^f(p_f, w, \theta_d) = \left[\theta_d (1-\tau)^{\alpha_d} \left(\frac{\alpha_d}{w} \right)^{\alpha_d} \left(\frac{\beta_d}{p_f} \right)^{1-\alpha_d} \right]^{\frac{1}{1-\alpha_d-\beta_d}} \qquad (60)
$$

The supply function for the downstream producers is defined by:

$$
y(\theta_d, w, p_f) = \left[\theta_d (1-\tau)^{\alpha_d} \left(\frac{\alpha_d}{w} \right)^{\alpha_d} \left(\frac{\beta_d}{p_f} \right)^{\beta_d} \right]^{\frac{1}{1-\alpha_d-\beta_d}} \qquad (61)
$$

and profits given by:

$$
\Pi_d^f(\theta_d, w, p_f) = (1-\tau)(1-\alpha_d-\beta_d) \left[\theta_d (1-\tau)^{\alpha_d} \left(\frac{\alpha_d}{w} \right)^{\alpha_d} \left(\frac{\beta_d}{p_f} \right)^{\beta_d} \right]^{\frac{1}{1-\alpha_d-\beta_d}} \qquad (62)
$$

Firms that decide to be informal but that need to constraint their size, will demand labor ($l^i_{d,c}$) and upstream goods ($x^i_{d,c}$) according to:

$$l^i_{d,c}(\bar{y}_d, w, \theta_d, p_i) = \left[\frac{\bar{y}_d}{\theta_d} \left(\frac{p_i \alpha_d}{w \beta_d} \right)^{\beta_d} \right]^{\frac{1}{\alpha_d + \beta_d}} \tag{63}$$

$$x^i_{d,c}(\bar{y}_d, w, \theta_d, p_i) = \left[\frac{\bar{y}_d}{\theta_d} \left(\frac{\beta_d w}{\alpha_d p_i} \right)^{\alpha_d} \right]^{\frac{1}{\alpha_d + \beta_d}} \tag{64}$$

Informal firms that are not constrained will demand labor and upstream goods according to the profit maximization solution in:

$$\Pi^i_{u,d}(\theta_d, w, p_i) = \max_{l, x_i} \theta_d l^{\alpha_d} x^{\beta_d} - wl - p_i x_i \tag{65}$$

the corresponding labor and upstream demands that solve this problem are given by:

$$l^i_{d,u}(p_i, w, \theta_d) = \left[\theta_d \left(\frac{\alpha_d}{w} \right)^{1-\beta_d} \left(\frac{\beta_d}{p_i} \right)^{\beta_d} \right]^{\frac{1}{1-\alpha_d-\beta_d}} \tag{66}$$

$$x^i_{d,u}(p_f, w, \theta_d) = \left[\theta_d \left(\frac{\alpha_d}{w} \right)^{\alpha_d} \left(\frac{\beta_d}{p_i} \right)^{1-\alpha_d} \right]^{\frac{1}{1-\alpha_d-\beta_d}} \tag{67}$$

the supply and profits of such firms will be given by:

$$y^i_{d,u}(\theta_d, w, p_i) = \left[\theta_d \left(\frac{\alpha_d}{w} \right)^{\alpha_d} \left(\frac{\beta_d}{p_i} \right)^{\beta_d} \right]^{\frac{1}{1-\alpha_d-\beta_d}} \tag{68}$$

$$\Pi^i_{d,u}(\theta_d, w, p_f) = (1 - \alpha_d - \beta_d) \left[\theta_d \left(\frac{\alpha_d}{w} \right)^{\alpha_d} \left(\frac{\beta_d}{p_i} \right)^{\beta_d} \right]^{\frac{1}{1-\alpha_d-\beta_d}} \tag{69}$$

Combining the problems of the unconstrained and the constrained informal firms, the solution of an informal firm in the downstream sector is characterized by the following demands:

$$l_d^i(p_i, w, \theta_d, \bar{y}) = \begin{cases} \left[\theta_d \left(\frac{\alpha_d}{w}\right)^{1-\beta_d} \left(\frac{\beta_d}{p_i}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} & \text{if } \theta < \underline{\theta_d}(\bar{y}, w, p_i) \\ \left[\frac{\bar{y}_d}{\theta_d} \left(\frac{\alpha_d p_i}{\beta_d w}\right)^{\beta_d}\right]^{\frac{1}{\alpha_d+\beta_d}} & \text{otherwise} \end{cases} \tag{70}$$

$$x_d^i(p_i, w, \theta_d, \bar{y}) = \begin{cases} \left[\theta_d \left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_i}\right)^{1-\alpha_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} & \text{if } \theta < \underline{\theta_d}(\bar{y}, w, p_i) \\ \left[\frac{\bar{y}_d}{\theta_d} \left(\frac{\beta_d w}{\alpha_d p_i}\right)^{\alpha_d}\right]^{\frac{1}{\alpha_d+\beta_d}} & \text{otherwise} \end{cases} \tag{71}$$

and the supply and profits given respectively by:

$$y_d^i(p_i, w, \theta_d, \bar{y}) = \begin{cases} \left[\theta_d \left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_i}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} & \text{if } \theta < \underline{\theta_d}(\bar{y}, w, p_i) \\ \bar{y}_d & \text{otherwise} \end{cases} \tag{72}$$

$$\Pi_d^i(p_i, w, \theta_d, \bar{y}) = \begin{cases} (1-\alpha_d-\beta_d) \left[\theta_d \left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_i}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} & \text{if } \theta < \underline{\theta_d}(\bar{y}, w, p_i) \\ \bar{y}_d - \left(\frac{\bar{y}_d}{\theta_d}\right)^{\frac{1}{\alpha_d+\beta_d}} \left[\left(w^{\alpha_d} p_i^{\beta_d}\right)^{\frac{1}{\alpha_d+\beta_d}} \left[\left(\frac{\alpha_d}{\beta_d}\right)^{\frac{\beta_d}{\alpha_d+\beta_d}} + \left(\frac{\beta_d}{\alpha_d}\right)^{\frac{\alpha_d}{\alpha_d+\beta_d}}\right]\right] & \text{otherwise} \end{cases} \tag{73}$$

Aggregate demand of the upstream informal good is given by

$$X_d^i(\bar{y}_d, w, p_i, p_f) = \int_{\theta_d^{min}}^{\underline{\theta}_d(\bar{y},w,p_i)} \left[\theta_d \left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_i}\right)^{1-\alpha_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} dF_d(\theta_d)$$
$$+ \int_{\underline{\theta}_d(\bar{y},w,p_i)}^{\hat{\theta}_d(p_i,p_f,w,\bar{y})} \left[\frac{\bar{y}_d}{\theta_d} \left(\frac{w\beta_d}{p_i\alpha_d}\right)^{\alpha_d}\right]^{\frac{1}{\alpha_d+\beta_d}} dF_d(\theta_d) \tag{74}$$

for the upstream formal good is

$$X_d^f(\bar{y}_d, w, p_i, p_f) = \int_{\hat{\theta}_d(p_i,p_f,w,\bar{y})}^{\theta_d^{max}} \left[\theta_d(1-\tau)^{\alpha_d} \left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_f}\right)^{1-\alpha_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} dF_d(\theta_d) \tag{75}$$

If we assume a Pareto distribution for θ_d such that the pdf is:

$$f(\theta_d; \gamma_d, \theta_{d,\min}) = \begin{cases} \frac{\gamma_d \theta_{d,\min}^{\gamma_d}}{\theta_d^{\gamma_d+1}} \text{ if } \theta_d \geq \theta_d^{\min} \\ \\ 0 \text{ otherwise} \end{cases} \tag{76}$$

The CDF is given by:

$$F(\theta_d; \gamma_d, \theta_d^{\min}) = \begin{cases} 0 \text{ if } \theta_d < \theta_{d,\min} \\ 1 - \left(\frac{\theta_d^{\min}}{\theta_d}\right)^{\gamma_d} \text{ otherwise} \end{cases} \tag{77}$$

The corresponding unconstrained informal demand is given by:

$$X_d^{i,u}(p_i, \bar{y}_d, w) = \left[\left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_i}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} \left(\frac{\gamma_d}{\kappa_d+1}\right) \left(\theta_d^{\min}\right)^{\gamma_d} \left(\underline{\theta}_d^{\kappa_d+1} - \left(\theta_d^{\min}\right)^{\kappa_d+1}\right) \tag{78}$$

with κ_d defined as follows:

$$\kappa_d = \frac{\alpha_d + \beta_d - \gamma_d(1 - \alpha_d - \beta_d)}{1 - \alpha_d - \beta_d} \tag{79}$$

The informal constrained demand is:

$$X_d^{i,c}(p_i, p_f, \bar{y}_d, w) = \int_{\underline{\theta}_d}^{\hat{\theta}_d(p_i, p_f, \bar{y}, w)} \left[\left(\frac{\bar{y}_d}{\theta_d}\right) \left(\frac{w\beta}{\alpha p_i}\right)^{\alpha_d}\right]^{\frac{1}{\alpha_d+\beta_d}} dF(\theta_d) \tag{80}$$

and under the assumption of a Pareto distribution, this demand becomes:

$$X_d^{i,c}(p_i, p_f, \bar{y}_d, w, \tau) = \left[\bar{y}_d \left(\frac{w\beta_d}{p_i \alpha_d}\right)^{\alpha_d}\right]^{\frac{1}{\alpha_d+\beta_d}} \left(\frac{\gamma_d}{c_d}\right) \left(\theta_d^{\min}\right)^{\gamma_d} \left(\hat{\theta}(p_i, p_f, w, \bar{y}, \tau)^{c_d} - \underline{\theta}_d^{c_d}\right) \tag{81}$$

with

$$c_d = \frac{1 - \gamma_d(\alpha_d + \beta_d)}{\alpha_d + \beta_d} \tag{82}$$

And the formal demand is given by:

$$X_d^f(p_i, p_f, \bar{y}_d, w, \tau) = \lim_{\tilde{\theta} \to \infty} \left[\left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_f}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} \left(\frac{\gamma_d}{\kappa_d}\right) \left(\theta_d^{\min}\right)^{\gamma_d} \left(\tilde{\theta}_d^{\kappa_d} - \left(\hat{\theta}_d(p_i, p_f, w, \bar{y}_d, \tau)\right)^{\kappa_d}\right) \tag{83}$$

which, under the assumption that $\alpha_d + \beta_d < 1$ and $\gamma_d > 1$ becomes

$$X_d^f(p_i, p_f, \bar{y}_d, w, \tau) = \left[\left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_f}\right)^{\beta_d} \right]^{\frac{1}{1-\alpha_d-\beta_d}} \left(\frac{\gamma_d}{\kappa_d}\right) (\theta_d^{\min})^{\gamma_d} \left(- \left(\hat{\theta}_d(p_i, p_f, w, \bar{y}_d, \tau) \right)^{\kappa_d} \right) \tag{84}$$

B.2 Equilibrium

Equilibrium in the upstream economy occurs as supply and demand equal in both, the formal and the informal market. Formally:

$$X_d^i(\bar{y}_d, w, p_i, p_f) = X_u^i(\bar{y}_u, w, p_i, p_f) \tag{85}$$

$$\begin{aligned}
&\underbrace{\left[\left(\frac{\alpha_d}{w}\right)^{\alpha_d} \left(\frac{\beta_d}{p_i}\right)^{\beta_d} \right]^{\frac{1}{1-\alpha_d-\beta_d}} \left(\frac{\gamma_d}{\kappa_d}\right) (\theta_d^{\min})^{\gamma_d} \left(\underline{\theta}_d^{\kappa_d} - (\theta_d^{\min})^{\kappa_d} \right)}_{\text{Informal unconstrained demand}} \\
&+ \underbrace{\left[\bar{y} \left(\frac{w\beta}{\alpha p_i}\right)^{\alpha} \right]^{\frac{1}{\alpha_d+\beta_d}} \left(\frac{\gamma_d}{c_d}\right) (\theta_d^{\min})^{\gamma_d} \left(\hat{\theta}(p_i, p_f, w, \bar{y}, \tau)^{c_d} - \underline{\theta}_d^{c_d} \right)}_{\text{Informal constrained demand}} = \\
&\underbrace{(\theta_u^{\min})^{\gamma_u} \left[\left(\frac{\alpha_u p_i}{w}\right)^{\frac{\alpha_u}{1-\alpha_u}} \left(\frac{\gamma_u(1-\alpha_u)}{\gamma_u(1-\alpha_u)-1}\right) \left((\theta_u^{\min})^{\frac{1-\alpha_u\gamma_u-\gamma_u}{1-\alpha_u}} - \underline{\theta}_u(\bar{y}_u)^{\frac{1-\alpha_u\gamma_u-\gamma_u}{1-\alpha_u}} \right) \right]}_{\text{Informal unconstrained supply}} \\
&+ \underbrace{\left[(\theta_u^{\min})^{\gamma_u} \bar{y}_u \left[\left(\frac{1}{\underline{\theta}_u(p_i, p_f, w, \bar{y}_u, \tau)}\right)^{\gamma_u} - \left(\frac{1}{\hat{\theta}_u(p_i, p_f, w, \bar{y}_u, \tau)}\right)^{\gamma_u} \right] \right]}_{\text{Informal constrained supply}}
\end{aligned} \tag{86}$$

In the formal market:

$$X_d^f(\bar{y}_d, w, p_i, p_f) = X_u^f(\bar{x}_u, w, p_i, p_f) \tag{87}$$

$$\underbrace{\left[\left(\frac{\alpha_d}{w}\right)^{\alpha_d}\left(\frac{\beta_d}{p_f}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}}\left(\frac{\gamma_d}{\kappa_d}\right)(\theta_d^{\min})^{\gamma_d}\left(-\left(\hat{\theta}_d(p_i,p_f,w,\bar{y}_d,\tau)\right)^{\kappa_d}\right)}_{\text{Formal demand}} =$$

$$\underbrace{\left(\frac{\alpha_u p_f}{w}\right)^{\frac{\alpha_u}{1-\alpha_u}}\left(\frac{\gamma_u(1-\alpha_u)}{\gamma_u(1-\alpha_u)-1}\right)\frac{(\theta_u^{\min})^{\gamma_u}}{\hat{\theta}_u(p_i,p_f,\bar{y},w,\tau)^{\gamma_u(1-\alpha)-1}}}_{\text{Formal supply}} \tag{88}$$

the solution to the equations 88 and 86 yield equilibrium prices $p_i^*(w,\bar{y}_d,\bar{x},\tau), p_f^*(w,\bar{y}_d,\bar{x},\tau)$. The total supply of the final good, at the equilibrium prices (p_i^*, p_f^*) is given by:

$$Y(\bar{y}_d,\bar{x},\tau) = \underbrace{\int_{\theta_d^{min}}^{\underline{\theta}_d}\left[\theta_d\left(\frac{\alpha_d}{w}\right)^{\alpha_d}\left(\frac{\beta_d}{p_i^*}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} dF_d(\theta_d)}_{\text{Unconstrained informal firms}} + \underbrace{\bar{y}\left[F(\hat{\theta}_d) - F(\underline{\theta}_d)\right]}_{\text{constrained informal firms}}$$

$$+ \underbrace{\int_{\hat{\theta}_d}^{\theta_d^{max}}\left[\theta_d\left(\frac{\alpha_d}{w}\right)^{\alpha_d}\left(\frac{\beta_d}{p_f^*}\right)^{\beta_d}\right]^{\frac{1}{1-\alpha_d-\beta_d}} dF_d(\theta_d)}_{\text{Formal firms}} \tag{89}$$

where we do not make explicit the dependence of equilibrium prices and productivity thresholds for ease of notation. Hasta acá revisé. Revisar el código y luego comptuar cosas.

B.3 Enforcement effects

In this section we analyze what are the effects on total output as a response to changes in enforcement rules. Let us consider first a change in enforcement in the downstream sector:

$$\frac{\partial Y(\bar{y}_d,\bar{x},\tau)}{\partial \bar{y}_d} = f_d(\underline{\theta}_d)\left[\frac{\partial y_d^i(p_i^*,w,\underline{\theta}_d,\bar{y})}{\partial p_i^*}\frac{\partial p_i^*}{\partial \bar{y}} + \frac{\partial y_d^i(p_i^*,w,\underline{\theta}_d,\bar{y})}{\partial \underline{\theta}_d}\frac{\partial \underline{\theta}_d}{\partial \bar{y}} + \frac{\partial y_d^i(p_i^*,w,\underline{\theta}_d,\bar{y})}{\partial \bar{y}}\right]$$

$$+\left[F(\hat{\theta}_d) - \underline{\theta}_d\right] + \bar{y}_d\left[f(\hat{\theta}_d)\frac{\partial \hat{\theta}_d}{\partial \bar{y}_d} + f(\underline{\theta}_d)\frac{\partial \underline{\theta}_d}{\partial \bar{y}_d}\right]$$

$$f_d(\hat{\theta}_d)\left[\frac{\partial y_d^f(p_f^*,w,\hat{\theta}_d,\bar{y})}{\partial p_i^*}\frac{\partial p_i^*}{\partial \bar{y}} + \frac{\partial y_d^f(p_f^*,w,\hat{\theta}_d,\bar{y})}{\partial \hat{\theta}_d}\frac{\partial \hat{\theta}_d}{\partial \bar{y}} + \frac{\partial y_d^f(p_f^*,w,\hat{\theta}_d,\bar{y})}{\partial \bar{y}}\right] \tag{90}$$

C Data

We use the following classification of firms using data from the Mexican statistical agency (INEGI), which asks the firms in each sector whether its clients are consumers or other firms.

Sector	**Intermediate vs final**
Agriculture	Intermediate
Mining	Intermediate
Electricity	Intermediate
Construction	Intermediate
Manufacturing	Intermediate
Wholesale	Final
Retail	Final
Transport	Intermediate
Massive media information	Final
Financial and insurance services	Final
Real estate services	Final
Professional, scientific, and technical services	Intermediate
Education services	Final
Health and social assistance services	Final
Cultural and sports services, and other recreational services	Final
Temporal lodging and food and beverage preparation services	Final
Other services except government activities	Intermediate
Legislative, governmental, justice activities, and international organizations	Final

D DePaula and Scheinkman Model

E Environment

The productive sector is composed by two types of firms: upstream firms, that produce an intermediate good used in the production of a final good, and downstream firms, that produce the final good. There is a continuum of entrepreneurs for each stage of production.

E.1 The Upstream Sector

The entrepreneurs that produce in the upstream sector differ on their managerial ability $\theta_u > 0$, where the distribution of abilities is given by $g_u(\cdot)$. A firm in the upstream sector with a manager with ability θ_u produces exactly θ_u units of the intermediate good. Firms in the upstream sector can choose to be formal and pay value-added tax (VAT) on their activity, or produce in the informal sector and avoid paying taxes, subject to the probability of being caught by the tax enforcement authorities and losing all profits. For simplicity, and following XXX, we assume that firms that produce below a threshold level $\bar{y}$ have a zero probability of being detected by the tax authorities, while those producing above $\bar{y}$ have a probability equal to one. In equilibrium, some firms will choose to produce in the informal sector, given the managerial ability of its entrepreneur. The profits of a formal firm in the upstream sector are:

$$\Pi^u_f(\theta_u) = (1-\tau)p_f\theta_u$$

where τ is the VAT, p_f is the price of the upstream good produced by the formal firm, and θ_u is the ability of the firm's manager. Similarly, the profits of an informal firm are:

$$\Pi^u_i(\theta_u) = p_i \cdot \min\{\theta_u, \bar{y}_u\}$$

where p_i is the price of the informal upstream good, and the firm has a size constraint, given by the probability of detection equal to one above $\bar{y}$; an entrepreneur that chooses to be informal will never produce above $\bar{y}$.

In equilibrium, for there to be firms producing in the informal sector, it must be the case that $p_i \geq (1-\tau)p_f$, such that there is a benefit from being informal. Note that firms with $\theta_u \leq \bar{y}$ will sell all of their production in the informal sector. However, if $p_i > (1-\tau)p_f$, there exists thresholds $\underline{\theta}_u$ and $\bar{\theta}_u$ such that, in equilibrium, there is a positive mass of firms $\theta_u \in [\underline{\theta}_u, \bar{\theta}_u]$ that will prefer to be informal and constraint their size, which represents an inefficiency in the economy (Garicano et al. 2016). The lower threshold is $\underline{\theta}_u = \bar{y}$, while the upper threshold $\bar{\theta}_u$ is given by the point in which the entrepreneurs are indifferent between being in the formal or informal sectors:

$$\bar{\theta}_u = \frac{\bar{y}_u p_i}{(1-\tau)p_f} \tag{91}$$

In this case, assuming a uniform distribution on abilities $\theta_u \sim U[0, \theta^u_{max}]$, the total amount of production inefficiently discarded by informal firms is $\frac{1}{2\theta^u_{max}}\left(\frac{\bar{y}_u p_i}{(1-\tau)p_f} - \bar{y}\right)^2$, which is a dead-weight loss.

a

The total supply of upstream formal and informal goods is determined, respectively, by:

$$\begin{aligned} y^i(p_i, p_f, \bar{y}) &= \frac{1}{\theta^u_{max}}\left(\frac{\bar{y}_u^2 p_i}{(1-\tau)p_f} - \left(\frac{\bar{y}p_i}{(1-\tau)p_f}\right)^2\right) \\ y^f(p_i, p_f, \bar{y}) &= \frac{1}{2\theta^u_{max}}\left((\theta^u_{max})^2 - \left(\frac{\bar{y}_u p_i}{(1-\tau)p_f}\right)^2\right) \end{aligned}$$

For a general density of skills $dF_u(\theta)$:

$$y_u^i(p_i, p_f, \bar{y}_u) = \int_{\theta_e^{min}}^{\bar{y}_u} \theta_e dF_u(\theta_e) + \int_{\bar{y}_u}^{\bar{\theta}_u} \bar{y}_u dF_u(\theta) \tag{92}$$

$$y_u^f(p_i, p_f, \bar{y}_u) = \int_{\bar{\theta}_u}^{\theta_e^{max}} \theta_e dF_u(\theta) \tag{93}$$

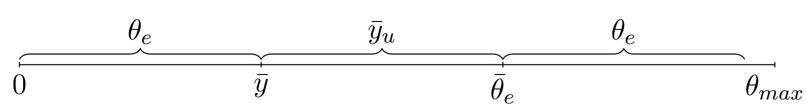

Figure 2: Production decisions for upstream firms depending on their entrepreneurial skill θ_e

E.2 The Downstream Sector

The second type of firms produce the downstream good, using as an input for production the managerial ability of the entrepreneur θ_d, distributed according to $g_d(\cdot)$, and the upstream good x, such that the total output is $f(\theta_d, x) = \theta_d x$. Firms in the downstream sector also choose whether to be formal or not, with the same consequences as firms in the upstream sector. The only difference is that the threshold for being undetected by the tax authorities applies to the amount of input used for production; firms that use $x \leq \bar{x}$ units of the intermediate good are not detected by the tax authorities, while those that use $x > \bar{x}$ are detected with probability equal to one. Moreover, even formal firms can buy their upstream input in the informal market. Formal firms that buy their input in informal markets do not receive the tax credit on the VAT paid, whereas those that buy in formal markets can deduct the VAT. The profits of a formal downstream firm are:

$$\Pi_f^d(\theta_d) = \max\{\max_x (1-\tau)\theta_d x - (1-\tau)p_f x, \quad \max_x (1-\tau)\theta_d x^\alpha - p_i x\}$$

In the case in which $p_i \geq (1-\tau)p_f$, producers of the downstream good in the formal sector will prefer to buy from formal producers, which will allow them to receive a credit for the VAT. The demand of formal firms for the intermediate good and the profits are, respectively:

$$x_f(\theta_d, p_f) = \left(\frac{\alpha\theta_d}{p_f}\right)^{\frac{1}{1-\alpha}}, \quad \Pi_f^d(\theta_d, p_f) = (1-\tau)\left[\theta_d\left(\frac{\alpha\theta}{p_f}\right)^{\frac{\alpha}{1-\alpha}} - p_f\left(\frac{\alpha\theta}{p_f}\right)^{\frac{1}{1-\alpha}}\right] = (1-\tau)\theta_d\left(\frac{\theta_d\alpha}{p_f}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha) \tag{94}$$

In the informal sector, downstream firms can also choose whether to buy the intermediate good from formal or informal firms. However, given that informal firms do not pay VAT, they cannot claim the tax credit, so the profits are:

$$\Pi_i^d(\theta_d) = \max\left\{\max_x \theta_d \cdot \min\{x,\bar{x}\}^\alpha - p_f \min\{x,\bar{x}\}, \quad \max_x \theta_d \cdot \min\{x,\bar{x}\}^\alpha - p_i \min\{x,\bar{x}\}\right\}$$

Note that informal firms will buy only from informal upstream producers so long as $p_i \leq p_f$, and only from formal producers otherwise. Therefore, for there to exist informal upstream producers, it must be the case that, in equilibrium, $(1-\tau)p_f \leq p_i \leq p_f$, and informal downstream producers buy only intermediate goods from informal upstream producers. The demand for intermediate good and profits are, respectively:

$$x_i(\theta_d, p_i) = \min\left\{\bar{x}, \left(\frac{\alpha\theta}{p_i}\right)^{\frac{1}{1-\alpha}}\right\}, \quad \Pi_i^d(\theta_d, p_i) = \theta_d \min\left\{\bar{x}, \left(\frac{\alpha\theta}{p_i}\right)^{\frac{1}{1-\alpha}}\right\}^{\alpha} - p_i \min\left\{\bar{x}, \left(\frac{\alpha\theta}{p_i}\right)^{\frac{1}{1-\alpha}}\right\} \tag{95}$$

The downstream firm decides to be formal whenever $\Pi_f^d(\theta_d, p_f) > \Pi_i^d(\theta_d, p_i)$. As shown in De Paula and Scheinkman, there exists a cut-off ability $\bar{\theta}_d$[4] such that entrepreneurs with $\theta_d < \bar{\theta}_d$ choose to be informal, whereas those with $\theta_d \geq \bar{\theta}_d$ are formal. The left panel in Figure 1 illustrates the profits and input demand of formal and informal buyers, where the dashed line is the value $\bar{\theta}_d$ at which the entrepreneur is indifferent between being formal or informal. Below the dashed line, entrepreneurs prefer to be informal and choose an input demand that is potentially constrained. Only those with ability above the dashed line prefer to be formal.

As in the upstream sector, there is a lower threshold $\underline{\theta}_d$ such that downstream firms with $\theta_d \in [\underline{\theta}_d, \bar{\theta}_d]$ will inefficiently produce exactly $\bar{x}$ to avoid being detected. The right panel in Figure 1 illustrates the effective input demand when there is VAT evasion in the economy. Those firms with $\theta < \bar{\theta}_d$ choose an input demand that allows them to remain non-monitored by the tax authorities, so they effectively pay no VAT. However, those above the dashed line have sufficiently large profits in the formal sector, so they choose to pay VAT. Therefore, the input demand has a discontinuity in $\bar{\theta}_d$, given that formal producers do not face the input constraint.

As in the intermediate good sector, we can characterize total demand for formal and informal upstream inputs, using equations (94) and (95):

$$\begin{aligned} x^i(p_i, p_f, \bar{x}) &= \int_{\theta\in[0,\underline{\theta}_d]} \left(\frac{\alpha\theta}{p_i}\right)^{\frac{1}{1-\alpha}} d\theta + \int_{\theta\in[\underline{\theta}_d,\bar{\theta}_d]} \bar{x} d\theta \\ x^f(p_i, p_f, \bar{x}) &= \int_{\theta\in[\bar{\theta}_d,\theta^d_{max}]} \left(\frac{\alpha\theta}{p_f}\right)^{\frac{1}{1-\alpha}} d\theta \end{aligned}$$

A seguir: 1. Cuantificar costos de misallocation. Cuál es el benchmark? full enforcement? si se elimina la posibilidad de informalidad? Una posibilidad puede ser simplemente eliminarla, si se hace enformernt perfecto. Otra puede ser si el enforcement es costoso, se gastan recursos en términos de enforcement. Obviamente eso va a ser difícil estimar esa función pero igual el resultado teórico va a decir hacer enforcement hasta que el retorno marginal no supere el costo, en los sectores con mayor visaje, algo así.

[4] The cut-off value $\bar{\theta}_d$ solves: $(1-\tau)\bar{\theta}_d^{\frac{1}{1-\alpha}}(\frac{1}{Pf})^{\frac{\alpha}{1-\alpha}}\left[\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right] = \theta_d\bar{x}^{\alpha} - P_i\bar{x}$, while ESTE ESTA MAL: $\underline{\theta}_d = \frac{p_i\bar{x}^{1-\alpha}}{\alpha}$.

E.3 Downstream sector if enforcement on output

Now let us consider the case when enforcement in the downstream is done via output. In such a case, a firm will be detected if its output exceeds a threshold $\bar{y}^d$. In such a case, the analysis for the formal downstream firm remains the same. The firm will consider to buy from an informal or a formal upstream firm by solving the following problem:

$$\Pi_f^d(\theta_d) = \max\{\max_x (1-\tau)\theta_d x^\alpha - (1-\tau)p_f x, \quad \max_x (1-\tau)\theta_d x^\alpha - p_i x\}$$

Formal firms will demand from formal downstream suppliers so long as $p_i \geq (1-\tau)p_f$. If this is the case, the problem of a downstream formal firm can be described according to:

$$\Pi_f^d(\theta_d) = (1-\tau)\max_x (\theta_d x^\alpha - p_f x) \tag{96}$$

The demand for the input is given by:

$$x_f(\theta_d, p_f) = \left(\frac{\alpha\theta_d}{p_f}\right)^{\frac{1}{1-\alpha}} \tag{97}$$

and the profits are given by

$$\Pi_f^d(\theta_d, p_f) = (1-\tau)\left(\frac{\theta_d}{p_f^\alpha}\right)^{\frac{1}{1-\alpha}}\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) \tag{98}$$

In the informal sector, downstream firms can also choose whether to buy the intermediate good from formal or informal firms. Downstream firms might be constrained due to the detection probability, or they might not depending on how productive they are. A firm that is constrained, will produce $\bar{y}^d$ as producing more will imply that it is detected. To produce $\bar{y}^d$, a firm must use a level of upstream input corresponding to $\bar{x}_i(\theta_d, \bar{y}^d) = \left(\frac{\bar{y}^d}{\theta_d}\right)^{\frac{1}{\alpha}}$. The solution to the problem of the downstream informal firm is:

$$x_i(\theta_d, p_i) = \min\left\{\left(\frac{\bar{y}^d}{\theta_d}\right)^{\frac{1}{\alpha}}, \left(\frac{\alpha\theta_d}{p_i}\right)^{\frac{1}{1-\alpha}}\right\} \tag{99}$$

Informal downstream firms will be constrained if their unconstrained production exceeds $\bar{y}^d$:

$$\left(\frac{\alpha\theta_d}{p_i}\right)^{\frac{\alpha}{1-\alpha}}\theta_d \geq \bar{y} \tag{100}$$

we call $\underline{\theta}_d$ the entrepreneurial ability threshold for informal downstream firms to be constrained:

$$\underline{\theta}_d = \left(\bar{y}^d\right)^{1-\alpha}\left(\frac{p_i}{\alpha}\right)^{\alpha} \tag{101}$$

Profits of informal downstream firms are then given by:

$$\Pi_i^d(\theta_d, \bar{y}^d, p_i) = \begin{cases} \left(\frac{\theta_d}{p_i^{\alpha}}\right)^{\frac{1}{1-\alpha}}\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) & \text{if } \theta_d < \underline{\theta_d} \\ \bar{y} - p_i\left(\frac{\bar{y}}{\theta_d}\right)^{1/\alpha} & \text{otherwise} \end{cases} \tag{102}$$

To analyze if a downstream firm decides to be formal or informal, we need to compare profits given in Equations 94 and 102. Consider first the case of an unconstrained firm. This firm will be informal whenever

$$(1-\tau)\left(\frac{\theta_d}{p_f^{\alpha}}\right)^{\frac{1}{1-\alpha}}\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) > \left(\frac{\theta_d}{p_i^{\alpha}}\right)^{\frac{1}{1-\alpha}}\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right)$$

which translates into unconstrained firms being informal whenever

$$(1-\tau)\, p_f^{\frac{-\alpha}{1-\alpha}} < p_i^{\frac{-\alpha}{1-\alpha}} \tag{103}$$

Equation 103, combined with the fact that formal firms downstream firms will only buy from formal suppliers when $p_f(1-\tau) < p_i$ means that the informal sector will only exist if:

$$p_f(1-\tau) < p_i < p_f\left(\frac{1}{1-\tau}\right)^{\frac{1-\alpha}{\alpha}} \tag{104}$$

A downstream firm with $\theta_d > \bar{\theta}_d$ will decide to be informal whenever

$$\bar{y}_d - p_i\left(\frac{\bar{y}_d}{\theta_d}\right)^{1/\alpha} > (1-\tau)\left(\frac{\theta_d}{p_f^{\alpha}}\right)^{\frac{1}{1-\alpha}}\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) \tag{105}$$

we call $\hat{\theta}_d(p_i, p_f, \alpha, \bar{y}_d)$ the level of entrepreneurial ability that makes a downstream firm indifferent between being informal or formal. That is, the value of θ that makes equation 105 hold with equality.

Total demand of informal goods from the upstream suppliers is equal to:

$$x_d^i(p_i, p_f, \bar{y}_d) = \int_0^{\underline{\theta}_d} \left(\frac{\alpha\theta}{p_i}\right)^{\frac{1}{1-\alpha}} dF_d(\theta) + \int_{\underline{\theta}_d}^{\hat{\theta}_d(p_i,p_f,\alpha,\bar{y}_d)} \left(\frac{\bar{y}^d}{\theta_d}\right)^{\frac{1}{\alpha}} dF_d(\theta) \tag{106}$$

$$x_d^f(p_i, p_f, \bar{y}_d) = \int_{\bar{\theta}_d}^{\theta_{max}^d} \left(\frac{\alpha\theta}{p_f}\right)^{\frac{1}{1-\alpha}} dF(\theta) \tag{107}$$

And total production of formal and informal goods is given by:

$$y_d^i(p_i, p_f, \bar{y}_d) = \int_0^{\underline{\theta}_d} \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \theta^{\frac{1}{1-\alpha}} dF_d(\theta) + \int_{\underline{\theta}_d}^{\hat{\theta}_d(p_i,p_f,\alpha,\bar{y}_d)} \left(\bar{y}^d\right) dF_d(\theta) \tag{108}$$

$$y_d^i(p_i, p_f, \bar{y}_d) = \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \int_0^{\underline{\theta}_d} \theta^{\frac{1}{1-\alpha}} dF_d(\theta) + y^d \left[F_d(\hat{\theta}_d(p_i, p_f, \alpha, \bar{y}_d)) - F_d(\underline{\theta}_d)\right] dF_d(\theta) \tag{109}$$

$$y_d^f(p_i, p_f, \bar{y}_d) = \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \int_{\hat{\theta}_d(p_i,p_f,\alpha,\bar{y}_d)}^{\theta_d^{max}} \theta^{\frac{1}{1-\alpha}} dF_d(\theta) \tag{110}$$

SI QUIERE CHECAR , ESTE THRESHOLD DEBERIA SER PARECIDO A LA CONDICION QUE ESTA EN EL PIE DE PAGINA DE LA PAGINA 3, SOLO QUE ALLA ESTA CON EL MONITORING THRESHOLD EN X Y NO EN Y..

Falta verificar que es una threshold rule y especificarla

E.3.1 Equilibrium if enforcement on output

In an equilibrium, given thresholds $(\bar{y}_d, \bar{y}_u)$, prices (p_i, p_f) should be such that markets clear. That is, demand and supply should be equal for both, formal and informal upstream goods. Equilibrium prices $(p_i^*(\bar{y}_d, \bar{y}_u), p_f^*(\bar{y}_d, \bar{y}_u)$ should be such that markets clear. Combining Equation 106 and 92, market clearing condition for the informal good is:

$$\int_0^{\underline{\theta}_d} \left(\frac{\alpha\theta}{p_i}\right)^{\frac{1}{1-\alpha}} dF_d(\theta) + \int_{\underline{\theta}_d}^{\hat{\theta}_d(p_i,p_f,\alpha,\bar{y}_d)} \left(\frac{\bar{y}^d}{\theta_d}\right)^{\frac{1}{\alpha}} dF_d(\theta) = \int_{\theta_e^{min}}^{\bar{y}_u} \theta_e dF_u(\theta_e) + \int_{\bar{y}_u}^{\bar{\theta}_u} \bar{y}_u dF_u(\theta) \quad (111)$$

and for the formal good:

$$\int_{\bar{\theta}_d}^{\theta^d_{max}} \left(\frac{\alpha\theta}{p_f}\right)^{\frac{1}{1-\alpha}} dF_d(\theta) = \int_{\bar{\theta}_u}^{\theta_e^{max}} \theta_e dF_u(\theta) \quad (112)$$

equilibrium prices $(p_i^*(\bar{y}_d, \bar{y}_u), p_f^*(\bar{y}_d, \bar{y}_u))$ solve for equations 111 and 112.

Total production in equilibrium as a function of $(\bar{y}_d, \bar{y}_u)$ can be found evaluating at the equilibrium prices.

$$Y(\bar{y}_d, \bar{y}_u) = \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \int_0^{\underline{\theta}_d} \theta^{\frac{1}{1-\alpha}} dF_d(\theta) + y^d \left[F_d(\hat{\theta}_d(p_i, p_f, \alpha, \bar{y}_d)) - F_d(\underline{\theta}_d)\right]$$

$$+ \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \int_{\hat{\theta}_d(p_i,p_f,\alpha,\bar{y}_d)}^{\theta_d^{max}} \theta^{\frac{1}{1-\alpha}} dF_d(\theta) \quad (113)$$

$$Y(\bar{y}_d, \bar{y}_u) = \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \left[\int_0^{\underline{\theta}_d\left(\bar{y}_d,\bar{y}_u,p_i^*(\bar{y}_d,\bar{y}_u),p_f^*(\bar{y}_d,\bar{y}_u)\right)} \theta^{\frac{1}{1-\alpha}} dF_d(\theta) + \int_{\hat{\theta}_d\left(\bar{y}_d,\bar{y}_u,p_i^*(\bar{y}_d,\bar{y}_u),p_f^*(\bar{y}_d,\bar{y}_u)\right)}^{\theta_d^{max}} \theta^{\frac{1}{1-\alpha}} dF_d(\theta)\right]$$

$$+ \bar{y}_d \left[F_d(\hat{\theta}_d\left(\bar{y}_d, \bar{y}_u, p_i^*(\bar{y}_d, \bar{y}_u), p_f^*(\bar{y}_d, \bar{y}_u)\right)) - F_d(\underline{\theta}_d\left(\bar{y}_d, \bar{y}_u, p_i^*(\bar{y}_d, \bar{y}_u), p_f^*(\bar{y}_d, \bar{y}_u)\right))\right] \quad (114)$$

E.3.2 Quantifying misallocation

We can quantify how much is lost due to informality as the amount of goods that would be produced if, given the same equilibrium prices, how much would constraint firms produce of the final output. This is a simple calculation of static misallocation that does not take into account changes in prices:

$$DL(\bar{y}_d, \bar{y}_u, p_i, p_f) = \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \left[\int_{\underline{\theta}_d}^{\hat{\theta}_d(p_i,p_f,\alpha,\bar{y}_d)} \theta^{\frac{1}{1-\alpha}} dF_d(\theta)\right]$$

$$- y^d \left[F_d(\hat{\theta}_d(p_i, p_f, \alpha, \bar{y}_d)) - F_d(\underline{\theta}_d)\right] \quad (115)$$

E.3.3 Enforcement strategies

In this subsection we ask how does output change as a response to a change in the enforcement strategy. Note that enforcement can be exerted either in the downstream $\bar{y}_d$ or the upstream $\bar{y}_u$ sector.

$$\frac{\partial Y(\bar{y}_d,\bar{y}_u)}{\partial \bar{y}_d} = \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \left[\underline{\theta}_d^{\frac{1}{1-\alpha}} f(\underline{\theta}_d)\frac{\partial \underline{\theta}_d}{\partial \bar{y}_d} - \hat{\theta}_d^{\frac{1}{1-\alpha}} f(\hat{\theta}_d)\frac{\partial \hat{\theta}_d}{\partial \bar{y}_d}\right] +$$

$$- \frac{1}{1-\alpha}\left(\frac{\alpha}{p_i}\right)^{\frac{1}{1-\alpha}} \frac{\partial p_i}{\partial \bar{y}_d}\left[\int_0^{\underline{\theta}_d} \theta^{\frac{1}{1-\alpha}} dF_d(\theta) + \int_{\hat{\theta}}^{\theta_d^{max}} \theta^{\frac{1}{1-\alpha}} dF_d(\theta)\right]$$

$$+ \bar{y}_d\left[F_d(\hat{\theta}_d)f(\hat{\theta}_d)\frac{\partial \hat{\theta}_d}{\partial \bar{y}_d} - F_d(\underline{\theta}_d)f(\underline{\theta}_d)\frac{\partial \underline{\theta}_d}{\partial \bar{y}_d}\right]$$

$$+ \left[F_d(\hat{\theta}_d) - F_d(\underline{\theta}_d)\right] \tag{116}$$

If enforcement is done in the upstream sector, the change in output is given by:

$$\frac{\partial Y(\bar{y}_d,\bar{y}_u)}{\partial \bar{y}_u} = \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \left[\underline{\theta}_d^{\frac{1}{1-\alpha}} f(\underline{\theta}_d)\frac{\partial \underline{\theta}_d}{\partial \bar{y}_u} - \hat{\theta}_d^{\frac{1}{1-\alpha}} f(\hat{\theta}_d)\frac{\partial \hat{\theta}_d}{\partial \bar{y}_d}\right] +$$

$$- \frac{1}{1-\alpha}\left(\frac{\alpha}{p_i}\right)^{\frac{1}{1-\alpha}} \frac{\partial p_i}{\partial \bar{y}_u}\left[\int_0^{\underline{\theta}_d} \theta^{\frac{1}{1-\alpha}} dF_d(\theta) + \int_{\hat{\theta}}^{\theta_d^{max}} \theta^{\frac{1}{1-\alpha}} dF_d(\theta)\right]$$

$$+ \bar{y}_d\left[F_d(\hat{\theta}_d)f(\hat{\theta}_d)\frac{\partial \hat{\theta}_d}{\partial \bar{y}_u} - F_d(\underline{\theta}_d)f(\underline{\theta}_d)\frac{\partial \underline{\theta}_d}{\partial \bar{y}_u}\right] \tag{117}$$

Otro objetivo puede ser maximizar revenue, podemos pensar en esa estrategia a ver también qué sucede pero primero empezar con esta más general.

E.4 Equilibrium

In equilibrium, given monitoring thresholds $\bar{x}$ and $\bar{y}$, prices p_i and p_f are determined such that supply is equal to demand of formal and informal goods. In the case of the informal market, equilibrium is characterized by:

$$x^i(p_i,p_f,\bar{x}) = y^i(p_i,p_f,\bar{y}) \tag{118}$$

and substituting for expressions in Equations XXX and YYY, we find that the equilibrium condition is given by:

$$\int_{\theta\in[0,\underline{\theta}_d]}\left(\frac{\alpha\theta}{p_i}\right)^{\frac{1}{1-\alpha}}dF(\theta)+\int_{\theta\in[\underline{\theta}_d,\bar{\theta}_d]}\bar{x}dF(\theta)=\frac{1}{\theta^u_{max}}\left(\frac{\bar{y}^2p_i}{(1-\tau)p_f}-\left(\frac{\bar{y}p_i}{(1-\tau)p_f}\right)^2/2\right) \tag{119}$$

which becomes:

$$\frac{1}{\theta^{max}_u}\left[\left(\frac{\alpha}{p_i}\right)^{\frac{1}{1-\alpha}}\frac{(1-\alpha)\underline{\theta}_d^{\frac{2-\alpha}{1-\alpha}}}{2-\alpha}+\bar{\theta}_d-\underline{\theta}_d\right]=\frac{1}{\theta^u_{max}}\left(\frac{\bar{y}^2p_i}{(1-\tau)p_f}-\left(\frac{\bar{y}p_i}{(1-\tau)p_f}\right)^2/2\right) \tag{120}$$

In the formal sector, the equilibrium condition is:

$$\tag{121}$$

$$x^f(p_i,p_f,\bar{x})=y^f(p_i,p_f,\bar{y}) \tag{122}$$

replacing the corresponding expressions:

$$\int_{\theta\in[\bar{\theta}_d,\theta^d_{max}]}\left(\frac{\alpha\theta}{p_f}\right)^{\frac{1}{1-\alpha}}d\theta=\frac{1}{2\theta^u_{max}}\left((\theta^u_{max})^2-\left(\frac{\bar{y}p_i}{(1-\tau)p_f}\right)^2\right) \tag{123}$$

$$\left(\frac{\alpha}{p_f}\right)^{\frac{1}{1-\alpha}}\left(\theta^d_{max}-\bar{\theta}_d\right)=\frac{1}{2\theta^u_{max}}\left((\theta^u_{max})^2-\left(\frac{\bar{y}p_i}{(1-\tau)p_f}\right)^2\right) \tag{124}$$

The values $p_i^*(\theta^d_{max},\bar{\theta}_d,\bar{y},\alpha,\tau,\underline{\theta}_d)$ and $p_f^*(\theta^d_{max},\bar{\theta}_d,\bar{y},\alpha,\tau,\underline{\theta}_d)$ solving for equations 124 and 120 characterize the equilibrium prices in this economy

E.5 Inefficiency and Misallocation

Esta sección probablemente borrar porque ya está quedando en la subsección quantifying misallocation y la de enforcement effects, quedan más general que asumir uniforme

Given the presence of informal markets, where firms avoid paying taxes, there are incentives for firms to remain small and produce smaller amounts than they would if they were part of formal markets. Therefore, there is an inefficiency, given that less output is produced than the socially optimal. We can quantify the foregone output due to this inefficiency as:

$$DL(\alpha,\tau,\theta_u^{max},p_i,p_f) = \int_{\theta\in[\underline{\theta}_d,\bar{\theta}_d]} \theta\bar{x}^{\alpha} - \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \theta^{\frac{1}{1-\alpha}} dF(\theta)$$

$$= \frac{\theta^2}{2}\bar{x}^{\alpha} - \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \frac{(1-\alpha)\theta^{\frac{2-\alpha}{1-\alpha}}}{2-\alpha}\Bigg|_{\underline{\theta}_d}^{\bar{\theta}_d} \tag{125}$$

Total output in this economy is given by:

$$O(\alpha,\theta_u^{max},\tau,\bar{x},\bar{y}) = \int_{\theta\in[0,\underline{\theta}_d]} \theta\left(\frac{\alpha\theta}{p_i}\right)^{\frac{\alpha}{1-\alpha}} dF(\theta) + \int_{\theta\in[\underline{\theta}_d,\bar{\theta}_d]} \theta\bar{x}^{\alpha}dF(\theta) + \int_{\theta\in[\bar{\theta}_d,\theta^d_{max}]} \theta\left(\frac{\alpha\theta}{p_f}\right)^{\frac{\alpha}{1-\alpha}} dF(\theta) \tag{126}$$

Which can be expressed as:

$$O(\alpha,\theta_u^{max},\tau,\bar{x},\bar{y}) = \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{1-\alpha}{2-\alpha}\right)\left[\underline{\theta}_d^{\alpha^*} + (\theta^d_{max})^{\alpha*} - \left(\bar{\theta}^d\right)^{\alpha*}\right] + \frac{\bar{x}^{\alpha}}{2}\left[\bar{\theta}^d - \underline{\theta}_d\right] \tag{127}$$

where $\alpha^* = \frac{2-\alpha}{1-\alpha}$Assume an enforcement technology is available for the government. This technology can decrease $\bar{x}$ or $\bar{y}$. A natural question is, where should the government enforce tax compliance in order to decrease the foregone output?

Luego le metemos más visaje. Si se supone que hay un costo en esta tecnología, por ejemplo, hasta qué punto vale la pena hacer el enforcement. Además creo que el caso que más sentido va a tener es si el enforcement se hace sobre el output del downstream y upstream.

Consider how output changes as a response in an increased enforcement in the downstream sector:

$$\frac{\partial O(\alpha,\theta_u^{max},\tau,\bar{x},\bar{y})}{\partial\bar{x}} = \alpha^{\frac{\alpha}{1-\alpha}}\left(\frac{1-\alpha}{2-\alpha}\right)\left[(p_i)^{-\frac{\alpha}{1-\alpha}} \times \left((2-\alpha)\left(\underline{\theta}_d^{\alpha*-1}\frac{\partial\underline{\theta}_d}{\partial\bar{x}} - \left(\bar{\theta}^d\right)^{\alpha*-1}\frac{\partial\bar{\theta}_d}{\partial\bar{x}}\right)\right)\right.$$

$$\left. -\frac{\alpha}{1-\alpha}p_i^{\frac{-1}{1-\alpha}}\frac{\partial p_i}{\bar{x}} \times \left(\underline{\theta}_d^{\alpha^*} + (\theta^d_{max})^{\alpha*} - \left(\bar{\theta}^d\right)^{\alpha*}\right)\right] +$$

$$\bar{x}^{\alpha}\left[\bar{\theta}^d - \underline{\theta}_d\right]\left[\frac{\partial\bar{\theta}^d}{\partial\bar{x}} - \frac{\partial\underline{\theta}_d}{\partial\bar{x}}\right] + \frac{\alpha\bar{x}^{\alpha-1}}{2}\left[\bar{\theta}_d - \underline{\theta}_d\right]^2 \tag{128}$$

The corresponding expression for enforcement in the upstream producers is:

$$\frac{\partial O(\alpha,\theta_u^{max},\tau,\bar{x},\bar{y})}{\partial \bar{y}} = \alpha^{\frac{\alpha}{1-\alpha}}\left(\frac{1-\alpha}{2-\alpha}\right)\left[(p_i)^{-\frac{\alpha}{1-\alpha}} \times \left((2-\alpha)\left(\underline{\theta}_d^{\alpha*-1}\frac{\partial \underline{\theta}_d}{\partial \bar{y}} - \left(\bar{\theta}^d\right)^{\alpha*-1}\frac{\partial \bar{\theta}_d}{\partial \bar{y}}\right)\right)\right.$$

$$\left. -\frac{\alpha}{1-\alpha}p_i^{\frac{-1}{1-\alpha}}\frac{\partial p_i}{\bar{y}} \times \left(\underline{\theta}_d^{\alpha^*} + (\theta_{max}^d)^{\alpha*} - \left(\bar{\theta}^d\right)^{\alpha*}\right)\right] +$$

$$\bar{x}^{\alpha}\left[\bar{\theta}^d - \underline{\theta}_d\right]\left[\frac{\partial \bar{\theta}^d}{\partial \bar{y}} - \frac{\partial \underline{\theta}_d}{\partial \bar{y}}\right] + \frac{\alpha \bar{x}^{\alpha-1}}{2}\frac{\partial \bar{x}}{\partial \bar{y}}\left[\bar{\theta}_d - \underline{\theta}_d\right]^2 \tag{129}$$

Esa expresión va a estar complicada. Sobretodo teniendo en cuenta que son precios de equilibrio y vamos a tener problema encontrando una expresión para los precios de equilibrio. Se puede poner ver cómo toda esta joda varía para diferentes niveles de $\bar{x}\bar{y}$ pero no son comparables. De entrada creo que no vamos a llegar a una expresión analítica de los precios de equilibrio pero se puede quizás trabajar con el análisis que está haciendo zarruk en la siguiente sección y tratar de decir algo de los efectos sobre los precios.

E.5.1 Inefficiency and misallocation when enforcement in downstream is done via output

Now let us consider the case when enforcement on both, downstream and upstream is done via total output generated. Total output can be characterized by the following expression:

$$O(\alpha,\theta_u^{max},\tau,\bar{y}_d,\bar{y}_u) = \int_0^{\underline{\theta}_d}\left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}}dF(\theta) + \int_{\underline{\theta}}^{\bar{\theta}}\bar{y}dF(\theta) + \int_{\bar{\theta}}^{\theta_d^{max}}\left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}}\theta^{\frac{\alpha}{1-\alpha}}dF(\theta) \tag{130}$$

The response in output as a consequence of changing the enforcement rule in the downstream sector is:

$$\frac{\partial O(\alpha,\theta_u^{max},\tau,\bar{y}_d,\bar{y}_u)}{\partial \bar{y}_d} = \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}}\underline{\theta}_d^{\frac{1}{1-\alpha}}f(\underline{\theta}_d)\frac{\partial \underline{\theta}_d}{\partial \bar{y}_d}$$

$$-\left(\frac{\alpha}{1-\alpha}\right)p_i^{\frac{-1}{1-\alpha}}\frac{\partial p_i}{\partial \bar{y}_d}\alpha^{\frac{\alpha}{1-\alpha}}\int_0^{\underline{\theta}_d}\theta^{\frac{1}{1-\alpha}}dF(\theta) + \bar{y}\left[f(\bar{\theta}^d)\frac{\partial \bar{\theta}^d}{\partial \bar{y}^d} + f(\underline{\theta}_d)\frac{\partial \underline{\theta}_d}{\partial \bar{y}_d}\right]$$

$$-\left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}}\bar{\theta}_d^{\frac{1}{1-\alpha}}f(\bar{\theta}_d)\frac{\partial \bar{\theta}^d}{\partial \bar{y}_d} - \left(\frac{\alpha}{1-\alpha}\right)p_i^{\frac{-1}{1-\alpha}}\alpha^{\frac{\alpha}{1-\alpha}}\frac{\partial p_i}{\partial \bar{y}_d}\int_{\bar{\theta}^d}^{\theta_{max}^d}\theta^{\frac{1}{1-\alpha}}dF(\theta) \tag{131}$$

The corresponding change in output as a consequence to changing the enforcement rule in the upstream sector is:

$$\frac{\partial O(\alpha, \theta_u^{max}, \tau, \bar{y}_d, \bar{y}_u)}{\partial \bar{y}_u} = \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \underline{\theta}_d^{\frac{1}{1-\alpha}} f(\underline{\theta}_d) \frac{\partial \underline{\theta}_d}{\partial \bar{y}_u}$$

$$- \left(\frac{\alpha}{1-\alpha}\right) p_i^{\frac{-1}{1-\alpha}} \frac{\partial p_i}{\partial \bar{y}_u} \alpha^{\frac{\alpha}{1-\alpha}} \int_0^{\underline{\theta}_d} \theta^{\frac{1}{1-\alpha}} dF(\theta)$$

$$- \left(\frac{\alpha}{p_i}\right)^{\frac{\alpha}{1-\alpha}} \bar{\theta}_d^{\frac{1}{1-\alpha}} f(\bar{\theta}_d) \frac{\partial \bar{\theta}^d}{\partial \bar{y}_u} - \left(\frac{\alpha}{1-\alpha}\right) p_i^{\frac{-1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \frac{\partial p_i}{\partial \bar{y}_u} \int_{\bar{\theta}^d}^{\theta^d_{max}} \theta^{\frac{1}{1-\alpha}} dF(\theta) \tag{132}$$

E.6 Ideas para pensar:

TO-DO: Mirar si en los datos hay esta constraint de tamaño entre los informales y una discontinuidad cuando se vuelven formales, como en la grafica 2. Pregunta: como miramos θ en los datos del censo? Encontrar algo asi en los datos serial genial.

Figure 5 muestra la cantidad de inputs que se estan dejando de comprar (y, por lo tanto, de output que se deja de producir) porque hay firmas que se quedan pegadas al constraint $\bar{X}$. Esto nos permite hablar de misallocation, porque hay muchas empresas con eficiencia relativamente alta, que estan produciendo con $\bar{X}$ para evadir impuestos, pero deberian estar produciendo mucho mas en el mercado legal.

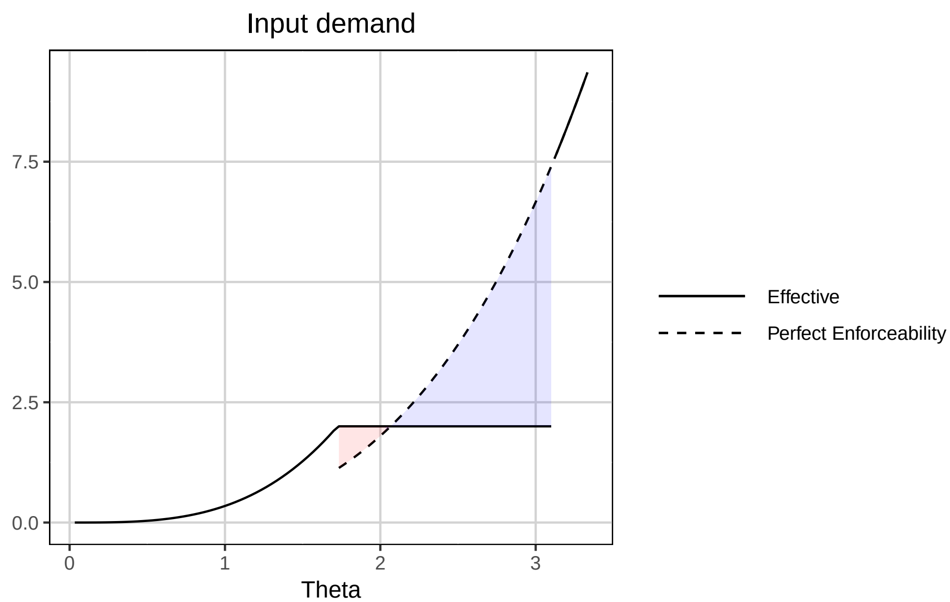

Figure 3: Input demand differences between an economy with evasion and one where the agents become formal when they hit the constraint $\bar{X}$.

F Third-Party Verification

What makes the VAT a more desirable tax over, for instance, a sales tax, is the third-party verification and paper trail left in the VAT collection. Once a producer reports having bought intermediate goods from a specific supplier, the tax collecting agency should expect the supplier to declare that sale. This paper trail generates a propagation mechanism along the production line where...

When the government increases the monitoring

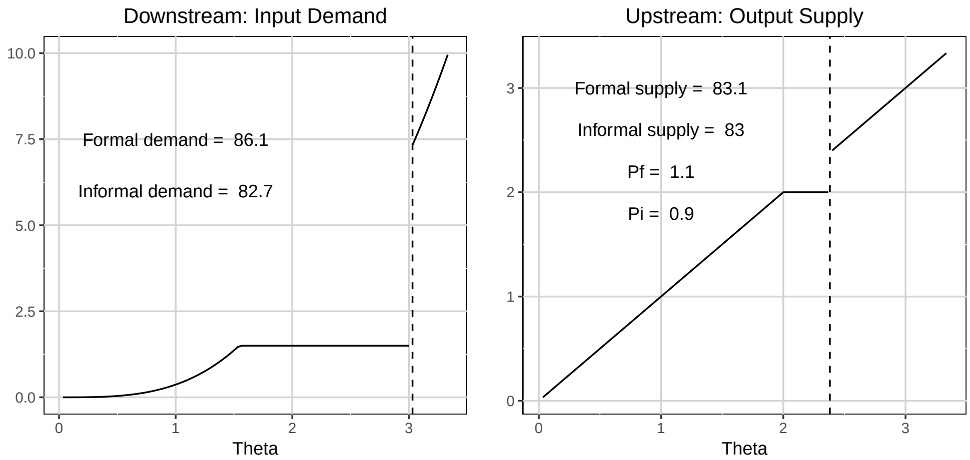

Figure 4: Supply (upstream) and demand (downstream) of formal and informal goods: $\bar{X} = 1.5$.

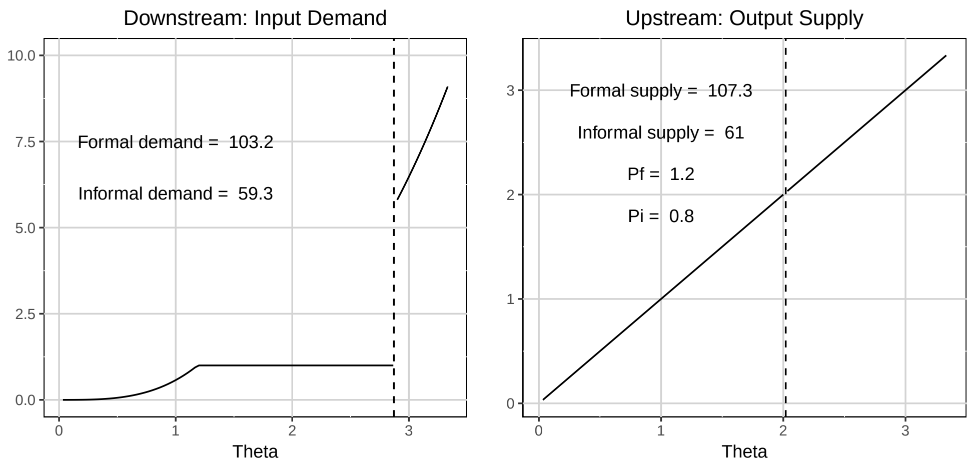

Figure 5: Supply (upstream) and demand (downstream) of formal and informal goods: $\bar{X} = 1$.

G Ideas aleatorias:

1. Decir que, de acuerdo con Pomeranz, la politica optima deberia ser monitorear en la parte de abajo de la cadena. Sin embargo, esto en el largo plazo, cuando precios se ajustan, no es cierto. Lo de ella pareciera ser algo de corto plazo.

2. Este modelo permite estudiar los efectos diferenciales de atacar la evasion en upstream o downstream, pero solamente a traves de general equilibrium effects. Es decir, dado que las firmas downstream compran el bien formal o informal en el mercado, y no a un mismo supplier, el "paper trail on transactions" del que habla Dina Pomeranz no se puede estudiar en este modelo. Podriamos modificarlo para que se adapte mas a eso.

3. Si no existieran costos de monitoreo, la solucion optima (del social planner), seria la de eliminar la evasion: esto reduciria misallocation

4. Con costos de monitoreo, la pregunta es: se debe atacar el upstream o el downstream? Los efectos podrian ser bien diferentes:

    (a) Atacando downstream, se reduce la demanda del bien informal y, por lo tanto, el precio. Esto hace que la cantidad demandada de input informal aumente (shifteando la curva de la parte informal en el grafico 2 hacia arriba)

    (b) Atacando el upstream, se reduce la oferta del bien informal, lo cual aumenta su precio. Esto reduce la cantidad demandada de input del bien informal (shifteando la curva de la parte informal en el grafico 2 hacia abajo => cerrando diferencia entre demanda de bien formal e informal)

Ideas aleatorias:

1. Si la elasticidad de sustitucion de los bienes formales e informales de consumo final no es 1, en equilibrio lo optimo no es eliminar toda la produccion informal.

2. Resultado adicional del paper puede ser. Pomeranz (2015) encuentra que hay una asimetría entre hacer enforcement en el upstream y en el downstream. Sin embargo, esto no tiene en cuenta efectos de equilibrio general. Cuando hay efectos de equilibrio general, puede haber efectos adicionales: enforcement en proveedor implica que aumenta oferta de proveedores formales, implica que cae el precio del insumo formal, implica que sube la demanda del insumo

formal, implica que hay mayor formalizacion en el bien final. Estos efectos *NO* los incluye pomeranz. Puede ser contribucion de nuestro paper. Hacer el enforcement en upstream y downstream puede tener efectos en el otro lado. Donde es más efectivo va a depender en ultimas de la elasticidad de la demanda y de la oferta. Considerar esto como contribucion. Explicar resultados de pomeranz implica no equilibrio general o corto plazo.

H Laffer Curve

I Misallocation

J Extension: Model with Labor

In this extension, we should include labor in the firm's problem. In this way, the choice of formality/informality, which will depend on